The Five Quadrable (Squarable) Lunes

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The story of the quadrature (squaring) of the lune spans 2500 years of history. It is a problem which was first studied by Greek mathematicians of the 5th century BCE but was not resolved until the middle of the 20th century CE when the Russian mathematicians Tschebotaryov and Dorodnov proved what had been suspected for some time, that only *five* such lunes existed. A lune is the crescent-like figure formed by intersecting two circles. To square a lune is to construct using only straight-edge and compass a square that has the same area. You can't just measure the area of the lune then construct a square with the same area; you have to *construct* the square from the lune.

Early Greek mathematicians were fascinated by three classical problems: doubling the cube (called the Delian Problem), trisecting the angle, and squaring the circle; that is constructing a square equal in area to a given circle. They also classified problems by the means used to solve them. According to Pappas of Alexandria (ca. 320 CE) p*lane* problems could be solved in the plane using only straight-edge and compass constructions, *solid* problems could be solved using conic sections and *linear* problems required the use of more complicated curves (Heath p. 218; Boyer p. 204). The means used to square the lune discussed here are *planar*: straight-edge and compass only!

The 5th century BCE Greek mathematician Hippocrates of Chios (not to be confused with the more famous *physician,* Hippocrates of Cos) is credited with finding *three* squarable lunes. According to a 6th century CE commentary on Artistotle by Philoponus, Hippocrates was originally a merchant who "fell in with a pirate ship and lost all his possessions" (Calinder p. 59) although other sources imply he was a not-so-clever merchant who was defrauded by corrupt tax-officials (van der Waerden p. 131). In any case he went to Athens to pursue his case against those who defrauded him, a process which apparently took a long time and whose outcome is not known. While in Athens he "consorted with philosophers" (Calinder p. 59) and becoming proficient in geometry used his new found talent to become a teacher. He is credited with writing a book on geometry (which was lost), finding a *non-planar* solution to the Delian problem of doubling the cube (which involved finding the solution to a double mean proportion) and, as already mentioned, squared the lune. In his book, van der Waerden (van der Waerden p. 132) suggests that Hippocrates' interest in squaring the lune might have stemmed from the similar problem of squaring the circle. Indeed the 3rd century CE writer Alexander of Aphrodisiensis credits Hippocrates with showing that if a certain *kind* of lune can be squared, then the circle can be squared (Dunham p. 20, van der Waerden p. 132).

Aside from its long history, the lune quadrature problem is interesting because of its ties with other famous problems in mathematics. Squaring the lune is an early attempt to find the area of a curved figure, something easily done today using integral calculus[1]. And because of the connection with squaring the circle, it has ties with find values for π.

Some Important Definitions and Preliminary Results

Before constructing the five quadrable lunes, we need to set down some definitions and results. Recall that a *lune* CGDF (figure 1) is the concave figure formed by the intersection of two circles (centers at A and B) usually of different diameters.

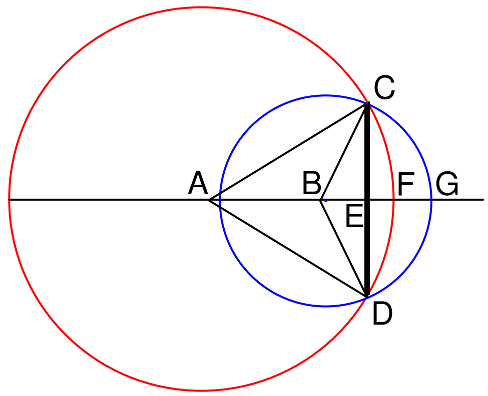

Figure 1

Given a central angle (e.g., $\angle CAD$) a *sector* is the area formed by the intersection of the two radii with the circumference of the circle. If θ is the measure of the central angle, the area of the sector is $\frac{\theta}{2}r^2$. If $\theta = 2\pi$ we have the familiar equation for the area of a circle.

A circular *segment* is the area between a chord which cuts the circle at two points and the circumference (CGDE). The base of the segment (CD) is the chord. Segments are *similar* if their central angles are equal. An important result used by Hippocrates (easily proved today) is that the areas of similar segments are to each other as the squares of their bases. Observe that a lune is the difference of two circular segments with a common base; lune CGDF = segment CGDE – segment CFDE.

Since we're restricted to straight-edge and compass constructions we need to be clear about what kinds of figures are constructible. It is important to point out that there is a close connection between constructible lines and a certain class of numbers, specifically a number is *constructible* if its magnitude which is represented by the length of a line segment can be constructed using only straight-edge and compass. For example, assuming a standard *unit* of length 1, given two lines of lengths a and b with a > b, it's

[1] See the article by Shenitzer and Stephans titled *The Evolution of Integration*; Am. Math. Monthly, Vol 101, No 1 Jan 1994, pp. 66-72.

not difficult to construct a line with length a+b or a–b simply by adding or subtracting length b to or from a.[2]

Multiplication and division constructions are done using similar right triangles. For example consider the two similar right triangles ABC and XBY in figure 2. Because of similarity

$$\overline{AB}/\overline{BC} = \overline{XB}/\overline{BY} \quad or \quad \overline{AB}\cdot\overline{BY} = \overline{XB}\cdot\overline{BC}$$

Let AB = a. If BC = 1 and BY = b then XB = ab. Alternately if BC = b and BY = 1 then XB = a/b. Thus magnitudes obtained by multiplication and division are constructible. .

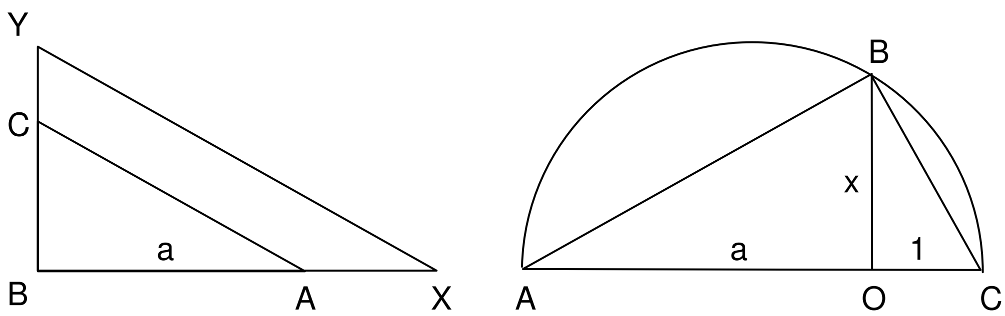

Figure 2

Taking the square root of magnitude a involves a semicircle (figure 2) whose diameter is a+1. If perpendicular OB is constructed where AO = a and OB = 1 then using similar right triangles AOB and BOC it can be shown that $x = \sqrt{a}$.

Thus magnitudes obtained by using the five operations of addition, subtraction, multiplication, division and square roots are constructible; for example numbers like $\frac{\sqrt{3-\sqrt{3}}}{2}$ or $\frac{\sqrt[4]{3}}{\sqrt{2}}$ in that lines can be constructed with these lengths. On the other hand it shown some cube roots, for example $\sqrt[3]{2}$, are not constructible.

Finally it should be point out that any rectilinear figure is is squarable or *quadrable*; that is, given any plane figure with straight sides, it is possible to construct a square with the same area. For example, given any rectangle, a square can be constructed with the same area (see Dunham p. 13). Given a triangle whose area is $\frac{1}{2}base\times height$, a rectangle can

[2] The constructions are given as Propositions I.2. and I.3 in Euclid's *Elements* which state "*From a given point … draw a straight line equal to a given line*" [Prop. I.2] and "*From the greater of two given straight lines .. cut off a part equal to the less*" [Prop I.3].

be constructed whose area equals the triangle and since the rectangle is quadrable, so is the triangle. Finally since any rectilinear figure can be partitioned into triangles each of which is quadrable and since given two squares, a third can be constructed equal to the sum of the other two (Pythagorean theorem), any rectilinear figure is quadrable.

The challenge now is to square a curved figure, like a lune.

The Three Classical Lunes of Hippocrates

Hippocrates of Chios is credited with finding three quadrable lunes which I refer to as the *isosceles triangle lune*, the *isosceles trapezoid lune*, and the *concave pentagon lune*. Their constructions are given below.

The Isosceles Triangle Lune

The easiest quadrable lune to construct and the one most often seen is based on an isosceles right triangle. See Figure 3 below.

1. On the base CE of a semicircle with center A construct the perpendicular bisector AD. Observe that CDE is an isosceles right triangle with $\overline{CE}:\overline{CD}=\sqrt{2}:1$

2. Let B be the midpoint of CD and construct a second semicircle with base CD and center B.

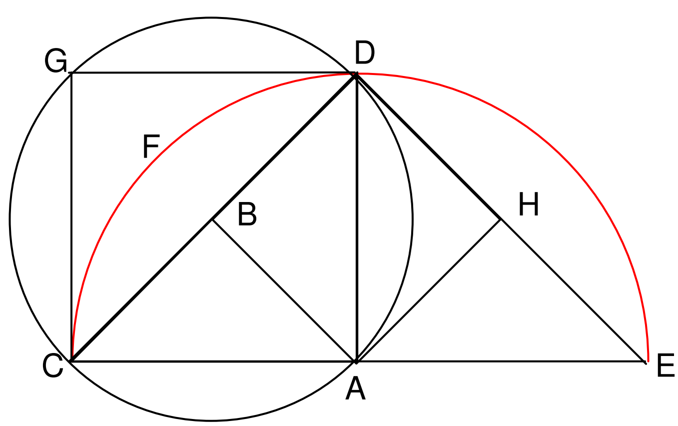

Figure 3

3. Since $\overline{CE}:\overline{CD}=\sqrt{2}:1$ the area of semi-circle CGD equals half the area of semi-circle CDE which is equal to the quarter circle ACFD.

4. Since the segment CFDB is common to both the quarter circle ACFD and the semicircle CGD, the lune CGDF equals the triangle ACD which also equals the *square* ABDH.

5. Alternately because the circular segments with bases CG and GD are *similar* to the circular segment with base CD and $\overline{CG}:\overline{GD}:\overline{CD}=1:1:\sqrt{2}$ it follows that lune CGDF is also equal to isosceles right triangle GCD.

The Isosceles Trapezoid Lune

The second quadrable lune is based on the construction of a constructible isosceles trapezoid CHKD the ratio of whose sides are $\overline{CD}:\overline{CH}:\overline{HK}:\overline{KD}=\sqrt{3}:1:1:1$. [3]

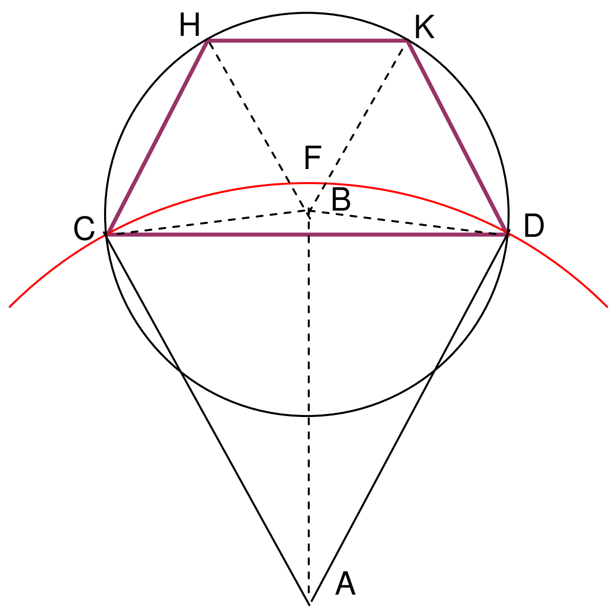

Figure 4

1. Since trapezoid CHKD is isosceles, it can be inscribed in a circle with center B[4].

2. Construct triangle CAD similar to triangle CBH and with center A then construct a circle thru points C and D. The circular segment with base CD is similar to circular segments with bases CH, HK, and KD.

3. Since $\overline{CD}:\overline{CH}:\overline{HK}:\overline{KD}=\sqrt{3}:1:1:1$, the circular segment with base CD is equal to the three circular segments with bases CH, HK, and KD. Thus the area of trapezoid CHKD equals lune CHKD.

4. Alternately it can be shown that the two sectors whose central angles are $\angle CBD$ and $\angle CAD$ are equal in area since $\angle CAD:\angle CBD=1:3$ and $\overline{CA}:\overline{CB}=\sqrt{3}:1$. After

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[3] Using right triangles it can be shown that the height of this trapezoid is $\sqrt[4]{3}/\sqrt{2}=\sqrt{\sqrt{3}/2}$ a constructible magnitude.

[4] The can be done by showing the perpendicular bisectors of CH, HK, and KD meet at a common point B.

subtracting the common area CFDB from both sectors, the resulting lune CHKD equals the kite-like quadrilateral ACBD.

The Concave Pentagon Lune

The third and last of the lunes that Hippocrates showed was constructible is based on a concave pentagon-like figure CFDKH (Figure 5) the ratio of whose sides are $\overline{CF}:\overline{FD}:\overline{DK}:\overline{KH}:\overline{HC}=\sqrt{3}:\sqrt{3}:\sqrt{2}:\sqrt{2}:\sqrt{2}$. Hippocrates' construction is more complicated than the other two and makes use of the *neusis* (verging or inclination) construction where a fixed length line segment is fitted between two curves in such a way that its extension passes through a given point.

1. Begin by constructing a semicircle with diameter JH and center K. Bisect KH at Q and erect a perpendicular QP.

2. Now comes the *neusis* construction. To quote Heath "*Let the straight line [DF] be so placed between [QP] and the circumference that it verges towards [H] … while its length is also such that the square on it is 1 ½ times the square of the radii*" (Heath p. 193). That is, for a point D on the semicircle, construct line DH such that if F is the point of intersection with QP then $\overline{DF}^2=\frac{3}{2}\overline{DK}^2$; the fixed length line segment DF when extended goes through H. [5]

3. From D, we construct a line parallel to the diameter JH and from H construct a line equal to $\overline{DK}$ which intersects this parallel at C. Thus CHKD is an isosceles trapezoid about which we circumscribe with a circle with center A.

[5] It's not exactly clear how to construct the line DH but we can construct a line FH such that if it is extended to D, then $\overline{DF}^2=\frac{3}{2}\overline{DK}^2$. Observe that if such a line existed, the isosceles triangles KFH and DKH are similar and thus it follows that $\overline{FH}/\overline{KH}=\overline{DK}/\overline{DH}$. Letting $a=\overline{DK}=\overline{KH}$ and $x=\overline{FH}$, since $\overline{DH}=\sqrt{\frac{3}{2}}a+x$ we obtain the quadratic $x^2+\sqrt{\frac{3}{2}}x-a^2=0$. Thus $\overline{FH}=x=\sqrt{\frac{3}{8}+a^2}-\sqrt{\frac{3}{8}}=\sqrt{\frac{3}{8}+\overline{DK}^2}-\sqrt{\frac{3}{8}}$ which is constructible using compass and straight-edge.

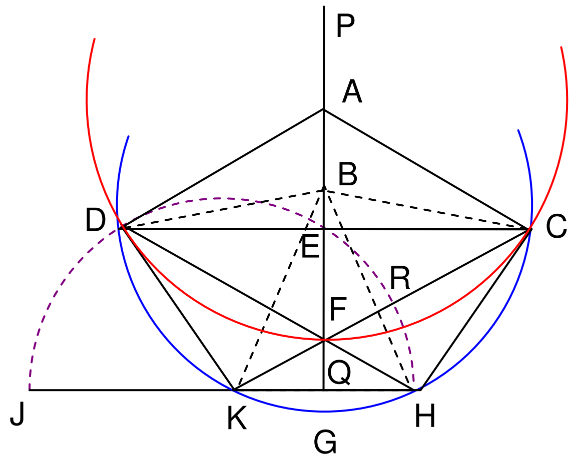

Figure 5

4. Next construct a second circle through the three points C, F and D with center B observing that $\overline{DF} = \overline{FC}$. Since $\overline{DF} : \overline{DK} = \sqrt{3} : \sqrt{2}$ then $\overline{CF} : \overline{FD} : \overline{DK} : \overline{KH} : \overline{HC} = \sqrt{3} : \sqrt{3} : \sqrt{2} : \sqrt{2} : \sqrt{2}$.

5. The next step is to show central angles $\angle FAC = \angle HBC$. If this is the case, then all five central angles are equal so the circular segments on bases CF, FD, DK, KH, and HC are similar. Since $\overline{CF} : \overline{FD} : \overline{DK} : \overline{KH} : \overline{HC} = \sqrt{3} : \sqrt{3} : \sqrt{2} : \sqrt{2} : \sqrt{2}$ lune CGDF equals the convex pentagon CHKDF.

6. To show $\angle FAC = \angle HBC$, for the corresponding isosceles triangles show base angles $\angle AFC$ and $\angle BHC$ are equal. This will be done using right triangles FEC and HRC, point R being where radius BH intersects chord KC at a right angle (this is easily proved using the congruency of triangles KRH and CRH). To show $\angle AFC = \angle BHC$ we consider their complements and show $\angle ECF = \angle HCR$.

7. To show $\angle ECF = \angle HCR$ note that $\angle HCK = \angle HKC$ (base angles of isosceles triangle KHC) and $\angle HKC = \angle ECK$ (alternate angles of parallel lines DC and KH).Thus $\angle ECF = \angle HCR$, $\angle AFC = \angle BHC$, and $\angle FAC = \angle HBC$.

8. Alternately we point out that since central angles $\angle CAD : \angle CBD = 2:3$ and radii $\overline{AC} : \overline{BC} = \sqrt{3} : \sqrt{2}$ the areas of sectors CAD and CBD are equal. Hence lune CGDF equals the area of the dart-like quadrilateral ACBD.

Finding All Quadrable Lunes: The Equation $\sqrt{m}\sin(\alpha) = \sin(m\alpha)$

Since the constructions for the quadrable lunes have become increasingly complex, we might ask if there is a general method for constructing all such lunes.

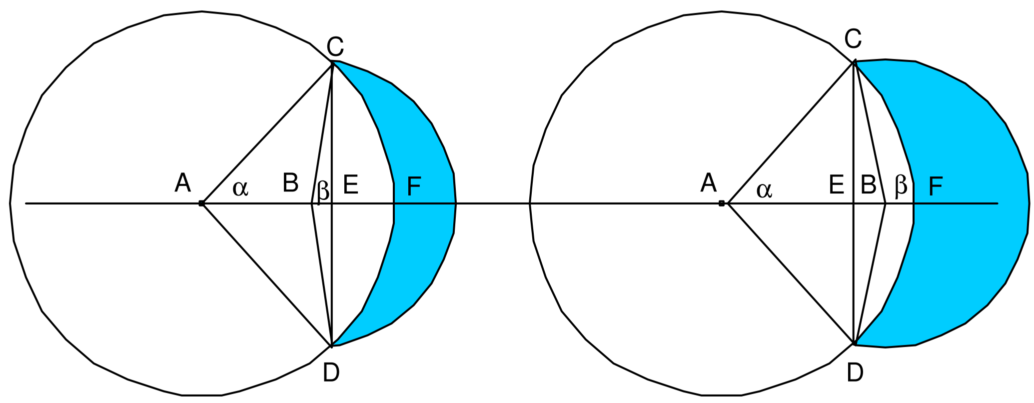

Figure 6

If a lune is the concave figure generated by the intersection of two circles with centers A and B, let CD be the line obtained from their intersection and E the point where CD intersects the line determined by centers A and B. Then either centers A and B are on one side of E (i.e ∠CBF = β is less than π/2) or E is between A and B (∠CBF = β is greater than π/2).

If we let 2α and 2β be the measures of the central angles ∠CAD and ∠CBD and r_α and r_β the corresponding radii of the two circles, it follows that the area of the lune is the difference of the two circular *segments* with base CD That is the area of the lune is

$$\beta \cdot r_\beta^2 \mp \cos(\beta)\sin(\beta) \cdot r_\beta^2 - \left(\alpha \cdot r_\alpha^2 - \cos(\alpha)\sin(\alpha) \cdot r_\alpha^2\right)$$

with the minus or plus depending on whether centers A and B are on the same or opposite sides of point E.

In the three lunes constructed above, the two *sectors* ACD and BCD were always equal in area. If we make this *simplifying assumption* that the two sectors have the same area, that is

$$\alpha \cdot r_\alpha^2 = \beta \cdot r_\beta^2$$

then the area of the lune reduces to

$$\cos(\alpha)\sin(\alpha) \cdot r_\alpha^2 \mp \cos(\beta)\sin(\beta) \cdot r_\beta^2$$

which in either case equals the area of the kite/dart-like quadrilateral ACBD.

If we let $\beta = m \cdot \alpha$ for some rational m, then the simplifying assumption $\alpha \cdot r_\alpha^2 = \beta \cdot r_\beta^2$ becomes $r_\beta^2 m\alpha = r_\alpha^2 \alpha \text{ or } \sqrt{m} = \frac{r_\alpha}{r_\beta}$.

We also observe that half of $\overline{CD}$, the common base for both circular segments, *unites* α and β, r_α and r_β; that is $\overline{CE} = r_\alpha \sin(\alpha) = r_\beta \sin(\beta)$. Substituting $m \cdot \alpha$ for β and $\sqrt{m}$ for $\frac{r_\alpha}{r_\beta}$ in this equation yields

$$\sqrt{m} \cdot \sin(\alpha) = \sin(m \cdot \alpha)$$

If we can solve this equation for sin(α) in such a way that sin(α) is constructible (that is sin(α) can be expressed using only the four arithmetic operations plus square roots), then it follows that $\alpha \cdot r_\alpha^2 = \beta \cdot r_\beta^2$ and hence the corresponding lune is equal to the area of the kite/dart-like quadrilateral ACBD.

Constructing the Lune

Suppose $\sqrt{m}\sin(\alpha) = \sin(m\alpha)$ has a constructible solution for m > 1, that is $\sin(\alpha)$ is constructible. To construct the quadrable lune, do the following.

Starting with line ℓ and constructible length $\sin(\alpha)$

1. Construct $\overline{CE} = \sin(\alpha) = \overline{DE}$ perpendicular to line ℓ at E.
2. Construct $\overline{AE} = 1 = \overline{AD}$ where A is a point on ℓ. Thus α is the measure of $\angle CAE = \angle DAE$.
3. Construct a circle with center A and radius $\overline{AC} = \overline{AD}$
4. Construct $\overline{CB} = \frac{1}{\sqrt{m}} = \overline{DB}$[6] noting there are two possibilities for B, the obtuse case which we label as B'. Let β be the measure of $\angle CBE = \angle DBE$ (acute case) and let β' be the measure of $\angle CB'G = \angle DB'G$ (obtuse case). Observe $\beta' = \pi - \beta$.
5. Construct a circle with center B (or B') and radius $\overline{CB} = \overline{DB}$ (or $\overline{CB'} = \overline{DB'}$).
6. Since $\sin(\beta) = \frac{\sin(\alpha)}{1/\sqrt{m}}$, then $\sin(\beta) = \sqrt{m} \cdot \sin(\alpha) = \sin(m \cdot \alpha)$. Since the same is true for β', either $\beta = m \cdot \alpha$ or $\beta' = m \cdot \alpha$.

[6] Since $\sqrt{m}\sin(\alpha) = \sin(m\alpha) \le 1$ it follows that $\sin(\alpha) \le \frac{1}{\sqrt{m}}$

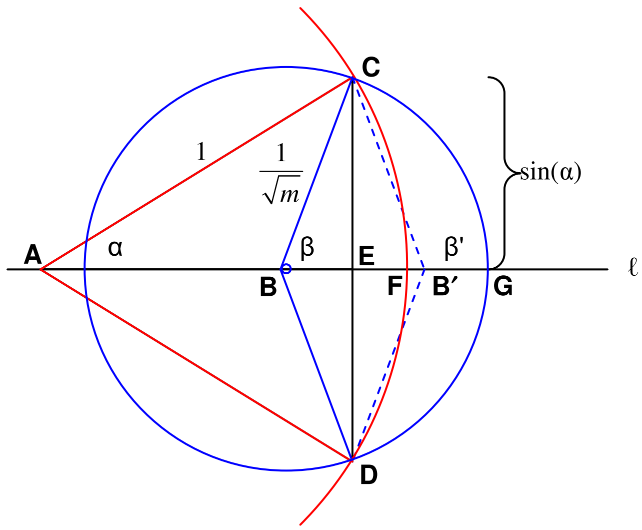

Figure 7

7. Without loss of generality let $m \cdot \alpha = \beta$. It then follows that sector CAD (central angle 2α, radius 1) and sector CBD (central angle 2β, radius $\frac{1}{\sqrt{m}}$) are equal in area. After deleting the common area, CFDB, the area of the lune CGDF remaining from sector CBD equals the area of the quadrilateral ACBD remaining from sector CAD. In the case of the isosceles right triangle lune, ACBD is an isosceles right triangle.

Thus lune CGDF is quadrable!

Two Other Quadrable Lunes

It turns out there are only five values of m for which the equation $\sqrt{m}\sin(\alpha) = \sin(m\alpha)$ has a constructible solution. The first three values, m = 2, m = 3 and m = 3/2, give the isosceles triangle, the isosceles trapezoid and the concave pentagon lunes found by Hippocrates[7]. The two other cases are m = 5 and m = 5/3.

m = 5: The Hexagon Lune

Rewrite $\sin(5\alpha) = \sqrt{5}\sin(\alpha)$ as $\sin(\alpha)\cos(4\alpha) + \sin(4\alpha)\cos(\alpha) = \sqrt{5}\sin(\alpha)$ which reduces to $\sin(\alpha)\left(\cos^2(2\alpha) - \sin^2(2\alpha)\right) + 2\sin(2\alpha)\cos(2\alpha)\cos(\alpha) = \sqrt{5}\sin(\alpha)$. Using

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[7] For m = 2 we obtain the equation $\sqrt{2}\sin(\alpha) = \sin(2\alpha) = 2sin(\alpha)\cos(\alpha)$ whose solution is $\cos(\alpha) = \frac{1}{\sqrt{2}}$ *or* $\sin(\alpha) = \frac{1}{\sqrt{2}}$. Thus sin(α) is constructible with $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{2}$. This is the isosceles right triangle lune. The details for m = 3 and m = 3/2 which yield the isosceles trapezoid and the concave pentagon lunes are given in the Appendix.

suitable substitutions this equation can be further reduced to the quadratic equation in $\sin^2(\alpha)$

$$16\sin^4(\alpha) - 20\sin^2(\alpha) + 5 - \sqrt{5} = 0$$

This can be solved for $\sin(\alpha) = \sqrt{\dfrac{5 \pm \sqrt{5 + 4\sqrt{5}}}{8}}$. Since the "plus" value results in a value greater than 1, only the "minus" value works. This makes $\alpha \approx 0.40909$ radians or 23.44°, $\beta \approx 2.04545$ radians or 117.196° and $\sin(\beta)/\sin(\alpha) = \sqrt{5}$.

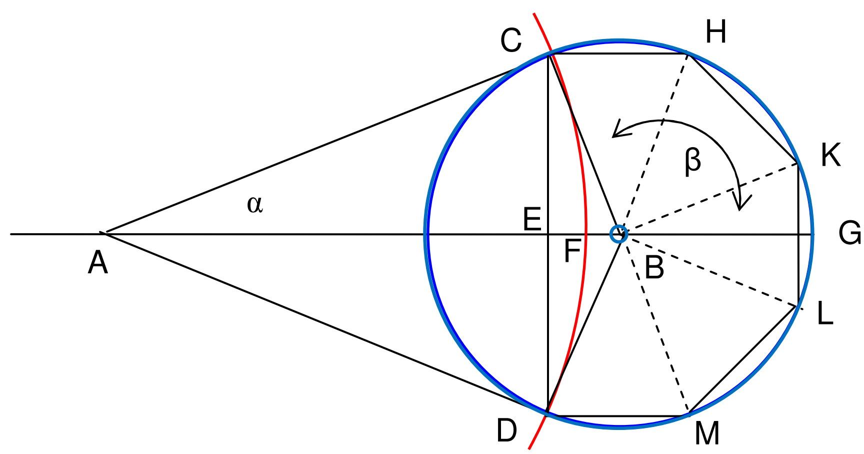

Figure 8

Since β is greater than 90°, point E is *between* A and B makes the lune look a bit like the isosceles trapezoid lune.

Figure 8 shows this lune with $\overline{CE} = \sin(\alpha) = \sqrt{\dfrac{5 - \sqrt{5 + 4\sqrt{5}}}{8}}$, $\overline{AC} = 1$, and $\overline{BC} = 1/\sqrt{5}$.
Lune CGDF is equal in area to the "kite-like" quadrilateral ACBD (because E is *between* A and B we add the area of figure CFDB).

We can also show that the area of the lune CGDF is equal to the area of the flattened hexagon CHKLMD where the ratio of the base $\overline{CD}$ to the other 5 sides is $\sqrt{5}:1:1:1:1:1$.

Starting with the figure obtained from the construction, construct ∠KBG = ∠ LBG equal to angle α = ∠CAE. Bisect angles CBK and LBD. Since β = 5α, ∠ CBH = ∠HBK =

$\angle$KBL = $\angle$LBM = $\angle$MBD = 2α so the circular segments on bases CH, HK, KL, LM, and MD are similar to the circular segment on base CD. Since $\overline{AC}:\overline{BC}$ as $\sqrt{5}:1$, $\overline{CD}:\overline{CH}:\overline{HK}:\overline{KL}:\overline{LM}:\overline{MD}=\sqrt{5}:1:1:1:1:1$ so it follows that hexagon CHKLMD equals lune CGDF.

$m=\frac{5}{3}$*: The Concave Octagon Lune*

The last lune for case m = 5/3 is the most complicated of the five. Starting with $\sin(\frac{5}{3}\alpha)=\sqrt{\frac{5}{3}}\sin(\alpha)$ use the substitution $\omega=\frac{\alpha}{3}$ to rewrite the equation as $\sin(5\omega)=\sqrt{\frac{5}{3}}\sin(3\omega)$.

Using suitable trig identities this can be rewritten as

$$\sin(\omega)\cos(4\omega)+\sin(4\omega)\cos(\omega)=\sqrt{\frac{5}{3}}\left(\sin(\omega)\cos(2\omega)+\sin(2\omega)\cos(\omega)\right)$$

which using double angle formulas for sine and cosine can be eventually reduced to a quadratic in $\sin^2(\omega)$, that is

$$16\sin^4(\omega)+\left(\frac{4}{3}\sqrt{15}-20\right)\sin^2(\omega)+\left(5-\sqrt{15}\right)=0$$

for which we obtain $\sin(\omega)=\sqrt{\frac{15-\sqrt{15}\pm\sqrt{60+6\sqrt{15}}}{24}}$, ugly but constructible. However only the "minus" expression satisfies $\sin(5\omega)=\sqrt{\frac{5}{3}}\sin(3\omega)$. Since $\omega=\frac{\alpha}{3}$ we have

$$\sin(\alpha)=\sin(3\omega)=\sin(\omega)\cos(2\omega)+\sin(2\omega)\cos(\omega)=\sin(\omega)(3-4\sin^2(\omega))$$

which results in $\sin(\alpha)=\sqrt{\frac{15-\sqrt{15}-\sqrt{60+6\sqrt{15}}}{24}}\left(\frac{3+\sqrt{15}+\sqrt{60+6\sqrt{15}}}{6}\right)$.

Using a calculator α ≈ 0.8793 radians or 50.38°, β ≈ 1.4655 radians or 83.97° and we can verify that $\frac{\sin(\beta)}{\sin(\alpha)}=\sqrt{\frac{5}{3}}$.

Since β is less than 90°, points A and B are not separated by E making the lune look a bit like concave pentagon lune.

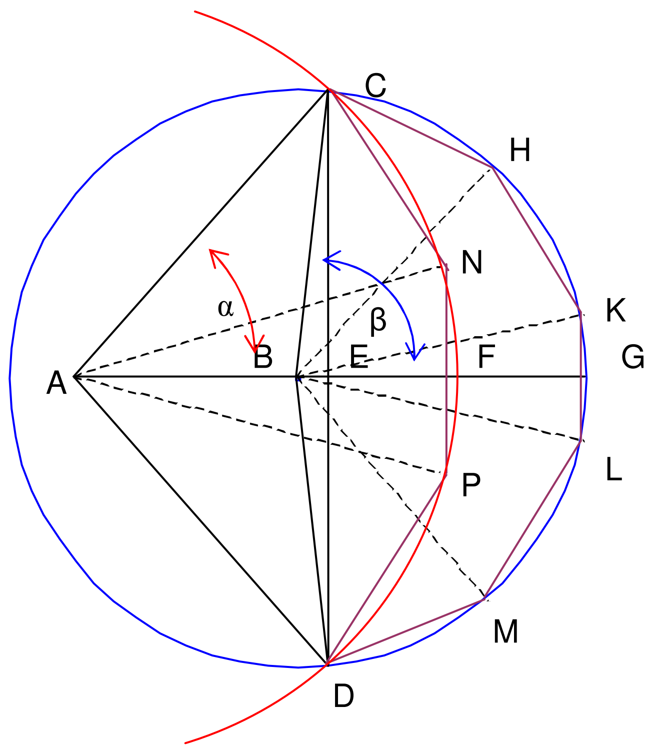

Figure 9

Figure 9 shows the lune where

$$\overline{CE} = \sin(\alpha) = \sqrt{\frac{15-\sqrt{15}-\sqrt{60+6\sqrt{15}}}{24}\left(\frac{3+\sqrt{15}+\sqrt{60+6\sqrt{15}}}{6}\right)}$$

$\overline{AC} = 1$ and $\overline{BC} = \sqrt{3/5}$. Since the two sectors ACD and BCD are equal in area, if we "subtract'" the area CBDF from both, it follows that lune CGDF is equal in area to the "dart-like" quadrilateral ACBD.

We can also show the lune is equal in area to the eight-sided concave figure CHKLMDPN. Since ∠CBG = β equals $\frac{5}{3}\alpha$, if we subtract angle α from it and bisect, we obtain the angle α/3. Construct angles∠ NAF = ∠ PAF = ∠KBG = ∠LBG = α/3. Bisect ∠CBK and ∠DBL. Thus ∠CAN = ∠NAP = ∠PAD = ∠CBH = ∠HBK = ∠KBL = ∠LBM = ∠MBD = $\frac{2}{3}\alpha$.

Consequently isosceles triangles CAN, NAP and PAD are mutually congruent as are isosceles triangles CBH, HBK, KBL, LBM, and MBD so the corresponding circular segments are similar. Since radii $\overline{AC}:\overline{BC} = \sqrt{5}:\sqrt{3}$ the circular segments on bases CN, NP and PD are to the circular segments on bases CH, HK, KL, LM, and MD as $\sqrt{5}:\sqrt{3}$.

Thus the area of the former three circular segments equals the area of the latter five circular segments so lune CGDF equals the concave octagon figure CHKLMDPN.

Conclusion and Summary

Are there any more lunes? By the end of the 19th century only five quadrable lunes were known. Squarable lunes were investigate by D. Bernouilli, Cramer, and Euler (Bashmakova & Smirnova p. 26) and Heath reports that all five cases were found by Martin Johan Wallenius in 1766 (Heath p. 200). In 1840 T. Clausen who rediscovered the five quadrable lunes suggested that there were only five. (Bashmakova & Smirnova p. 26). However it wasn't until 1934 when the Russian mathematician Tschebatorev, using Galois theory, came close to a solution which his student Dorodnov completed in 1947. Their proof that only five lunes are constructible (Shenitzer p. 646) brought to a close the quest for quadrable lunes.

To bring the paper full circle we end with a final construction attributed to Hippocrates that showed that if a certain lune were quadrable then the circle could be squared.

Start with semi-circles with bases AB and CD where $\overline{CD}$ is twice $\overline{AB}$. Inscribe within the second semi-circle CD an isosceles trapezoid CEFD made up of three equilateral triangles. On the three smaller sides construct three semi-circle with bases CH, HK, and DK.

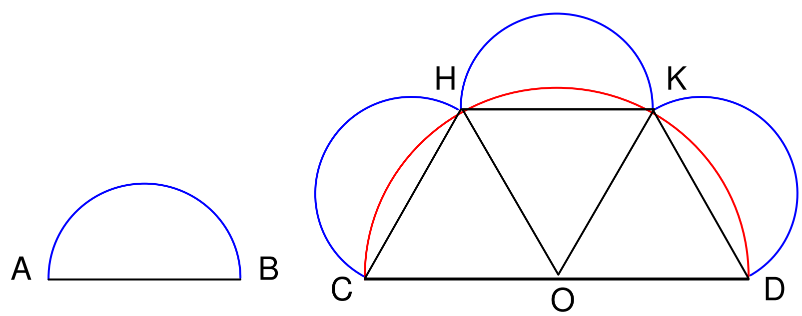

Figure 10

Since $\overline{CD}:\overline{CH}:\overline{HK}:\overline{FD} = 2:1:1:1$ it follows that the semi-circle on CD equals the four semi-circles on AB, CH, HK, and KD. It also follows that semi-circle on AB equals the trapezoid CHKD minus the three lunes on CH, HK, and KD.

Now *if* those three lunes formed by the intersection of the semi-circle on CD and the semi-circles in CH, HK, and KD were *quadrable*, then the semi-circle on AB is squarable! Hence the circle with diameter AB is squarable![8]. However, in 1882 C. L. F. Lindemann

[8] In his book *Journey Through Genius* [p. 20] Dunham points out that some ancient mathematicians believed that with this construction Hippocrates claimed that he had squared the circle. However Dunham explains that a mathematician of Hippocrates' caliber would hardly make this kind of error!

showed that π was transcendental. Thus it follows that the circle is not quadrable and hence the above lunes are not quadrable (Dunham p. 25).

In his book *Journey Through Genius*, William Dunham talks about "truly great theorems" (Dunham p. 285). He mentions three characteristics of great theorems which he attributes to G.H. Hardy: "*economy*, *inevitability*, and *unexpectedness*." Dunham points out quite correctly that there is an *aesthetics* to good mathematics as seen in the elegance and beauty of the logic of a proof, the inevitability and clarity of reasoning which makes the reader say upon conclusion "*of course this follows - how obvious*" .

But there is another dimension to good mathematics; it is the historical sweep of a good problem, the refinement of an original proof, the unexpected corollaries obtained later on, the fact that as mathematical notation and understanding grew and matured, that which was once difficult and obscure is made easy and transparent. While one admires the original pioneers for discoveries made under difficult conditions, one also appreciates the tools and notation of modern mathematics that now makes these discoveries "easily" accessible.

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Appendix – Calculations for cases m = 3 and m = 3/2

For m = 3 we obtain the equation $\sin(3\alpha) = \sqrt{3}\sin(\alpha)$. Using the sum and double angle identities for sine and cosine we obtain $\sin(\alpha)\cos(2\alpha) + \sin(2\alpha)\cos(\alpha) = \sqrt{3}\sin(\alpha)$ which reduces to $4\sin^2(\alpha) - \left(3+\sqrt{3}\right) = 0$ for which we obtain the solution $\sin(\alpha) = \dfrac{\sqrt{3-\sqrt{3}}}{2}$, a value which is easily constructible. Note that $\alpha \approx 0.5980$ radians or 34.26°, $\beta \approx 1.79409$ radians or 102.79° and using a calculator it is easily to check that $\sin(\beta)/\sin(\alpha) = \sqrt{3}$. This is the isosceles trapezoid lune.

For m = 3/2., starting with $\sin\left(\frac{3}{2}\alpha\right) = \sqrt{\frac{3}{2}}\sin(\alpha)$ use the substitution $\omega = \dfrac{\alpha}{2}$ to obtain the equation $\sin(3\omega) = \sqrt{\frac{3}{2}}\sin(2\omega)$ which can be rewritten as $\sin(\omega)\cos(2\omega) + \sin(2\omega)\cos(\omega) = \sqrt{6}\sin(\omega)\cos(\omega)$. This yields a quadratic in $\cos(\omega)$

$$4\cos^2(\omega) - \sqrt{6}\cos(\omega) - 1 = 0$$

Thus $\cos(\omega) = \dfrac{\sqrt{6}\pm\sqrt{22}}{8} = \dfrac{\sqrt{7\pm\sqrt{33}}}{4}$ and $\sin(\omega) = \dfrac{\sqrt{9\mp\sqrt{33}}}{4}$. Since $\alpha = 2\omega$,

$$\sin(\alpha) = \sin(2\omega) = 2\sin(\omega)\cos(\omega) = 2\frac{\sqrt{9\mp\sqrt{33}}}{4}\frac{\sqrt{7\pm\sqrt{33}}}{4} = \frac{\sqrt{30\pm 2\sqrt{33}}}{8}$$

Only the $\sin(\alpha) = \frac{\sqrt{30 + 2\sqrt{33}}}{8}$ solution works as a value which satisfies original equation. Thus $\alpha \approx 0.9359$ radians or 53.62° and $\beta \approx 1.4039$ radians or 80.44°and $\sin(\beta)/\sin(\alpha) = \sqrt{3/2} = \sqrt{3}/\sqrt{2}$). This is the concave pentagon lune.

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