Comp 260: Computational Models and Methods

Solutions to Dynamical Systems – Finding a “closed solution”

Method of Conjecture

Linear Dynamical Systems:

\[ a_{n+1} = r \cdot a_n \]

Non-Linear Systems:

\[ a_{n+1} = r (1 - a_n) a_n \]

Method of Conjecture (a.k.a. Guess & Check)

Example 1 – Compound Interest:

Observe a pattern – generate a table of values

\[ a_0 = 1000 \]

\[ a_1 = (0.01)(1000) = 10.01 \]

\[ a_2 = (0.01)(1000) = 10.01 \]

Conjecture a form of the solution:

\[ a_n = (1.01)^n a_0 \]

Test the conjecture

Accept or Reject

Dynamical System Examples of the Form: \( a_{n+1} = r \cdot a_n + b \)

Example 3: Digoxin

\[ a_{n+1} = 0.5 a_n + 0.1 \]

\[ a[0] = 0.3 \]

Example 4: Annuity

\[ a_{n+1} = 1.01 a_n - 1000 \]

Example 5: Checking Account

\[ a_{n+1} = a_n - 300 \]

Theorem 3: The solution to \( a_{n+1} = r \cdot a_n + b \) for \( r \neq 1 \) is

\[ a_n = \frac{b}{1 - r} + c \]

for some constant \( c \) which depends on the initial conditions.

Proof by Conjecture

Example 6: For the annuity in Example 4 what should the initial investment be to deplete the annuity in 20 years.

Solve: \( a_{20} = 0 = (1.01)^{20} c + 100000 \) to obtain

\[ c = -100000/(1.01)^{20} = -9180.58 \]

Thus \( a_0 = (1.01)^{20} c + 100000 = -9180.58 + 100000 = 90819.42 \)
For Wednesday
Read Ch. 3.4: Systems of Difference Equations
Ch. 2. Intro & 2.1: Mathematical Models

Verhulst Population Dynamics Equation
R is relative rate of population growth: 
\[ R = \frac{P_{n+1} - P_n}{P_n} \]  (1)
Therefore unrestricted growth is 
\[ p_{\infty} = (1 + R)^t_p_0 \]  (2)
If X is the limiting value or carrying capacity normalize to 1 and
the growth rate R is a linear function of the distance between p_n
and 1 then 
\[ R = c (1 - p_n) \] for some constant c; that is
\[ R = c (1 - p_n) \]  (3)
Therefore combining (1) and (3) yields
\[ p_{\infty} = (1 + c) p_n - cp_n^2 = p_n + c (1 - p_n) p_n \]