Linear Programming – The Simplex Method
Developed by George Dantzig

Starting at a known extreme-point moved to an adjacent intersection point that passes

**Optimality test:** shows whether an intersection point corresponds to a value of the objective function that is better

**Feasibility test:** determines if the intersection point is feasible

More computationally efficient than checking feasibility & optimality of all intersection points

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**Example – Carpenter’s Problem - 1**

Start with Objective Function \( 25x_1 + 30x_2 \)
and constraints

\[
\begin{align*}
20x_1 + 30x_2 & \leq 690 \\
5x_1 + 4x_2 & \leq 120 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Add Objective Function constraint (assuming all variables \( \geq 0 \))

\[
\begin{align*}
20x_1 + 30x_2 & \leq 690 \\
5x_1 + 4x_2 & \leq 120 \\
-25x_1 - 30x_2 & \leq 0
\end{align*}
\]

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**Feasibility Area: Six Intersection Points**

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**Example – Carpenter’s Problem - 2**

Add slack variables

\[
\begin{align*}
20x_1 + 30x_2 + y_1 & = 690 \\
5x_1 + 4x_2 + y_2 & = 120 \\
-25x_1 - 30x_2 + y_3 & = 0
\end{align*}
\]

Convert to augmented system (3 equations in 5 unknowns)

\[
\begin{align*}
20x_1 + 30x_2 + y_1 + 0x_3 + 0z & = 690 \\
5x_1 + 4x_2 + y_2 + 0x_3 + 0z & = 120 \\
-25x_1 - 30x_2 + 0y_1 + 0y_2 + z & = 0
\end{align*}
\]

Note that \( z \) is the value of the objective function

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**Tableau 0**

\[
\begin{array}{cccccc}
& x_1 & x_2 & y_1 & y_2 & z & \text{RHS} \\
20 & 30 & 1 & 0 & 0 & 690 \\
5 & 4 & 0 & 1 & 0 & 120 \\
-25 & -30 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Dependent variables \((y_1, y_2, z)\)

Independent variables \(x_1 + x_2 = 0\)

Extreme Point \((x_1, x_2) = (0, 0)\)

Value of Objective Function: \( z = 0 \)

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**Optimality Test (choosing entering variable \( x \neq 0 \)):** Choose the variable in the objective function equation that if not zeroed out, best improves the current objective value of \( z \) (generally the variable with an negative coefficient whose absolute value is largest).

**Feasibility Test (choosing exiting variable = 0):** Make one of the \( y \) variables the new entering variable. Must be chosen so that all variables \( \geq 0 \).

Divide RHS by coefficient of entering variable. Choose \( y_i \) with smallest ratio. The ratios represent the value of the entering variable would obtain if corresponding exiting variable were assigned zero.

Smallest positive ratio used to determine exiting variable so no variables are negative
Optimality Test: Choose $x_2$ as entering variable

Feasibility Test: Choose $y_1$ as exiting variable

Tableaux 1

\[
\begin{align*}
20x_1 + 30x_2 + y_1 + 0y_2 + 0z &= 690 \\
5x_1 + 4x_2 + 0y_1 + y_2 + 0z &= 120 \\
-25x_1 - 30x_2 + 0y_1 + 0y_2 + z &= 0
\end{align*}
\]

\[
\begin{array}{cccccc|c}
 x_1 & x_2 & y_1 & y_2 & z & RHS & Ratio \\
 20 & 30 & 1 & 0 & 0 & 690 & 690/30 = 23 \\
 5 & 4 & 0 & 1 & 0 & 120 & 120/4 = 30 \\
 -25 & -30 & 0 & 0 & 1 & 0 & 690/30 = 23 \\
\end{array}
\]

Note: if $y_1 = 0$

then $x_2 = 23$

and $y_2 = 28$

Note: if $y_2 = 0$

then $x_1 = 30$

and $y_1 < 0$

Dependent variables \{x_2, y_2, z\}

Independent variables: $y_1 = x_1 = 0$

Extreme Point \((x_1, x_2) = (0, 23)\)

Value of Objective Function: $z = 690$

Pivot – Eliminate $x_1$ from rows 1 and 3

\[
\begin{align*}
\frac{2}{3}x_1 + x_2 + \frac{1}{30}y_1 + 0y_2 + 0z &= 23 \\
\frac{7}{3}x_1 + 0x_2 - \frac{7}{15}y_1 + y_2 + 0z &= 28 \\
-5x_1 + 0x_2 + 0y_1 + 0y_2 + z &= 690
\end{align*}
\]

\[
\begin{array}{cccccc|c}
 x_1 & x_2 & y_1 & y_2 & z & RHS & Ratio \\
 2/3 & 1 & 1/30 & 0 & 0 & 23 & 23/(2/3) = 34.5 \\
 7/3 & 0 & -2/15 & 1 & 0 & 28 & 28/(7/3) = 12 \\
 -5 & 0 & 1 & 0 & 1 & 690 & 690/30 = 23 \\
\end{array}
\]

Tableaux 2 (Final)

\[
\begin{align*}
0x_1 + x_2 + \frac{1}{14}y_1 - \frac{1}{7}y_2 + 0z &= 15 \\
x_1 + 0x_2 + \frac{1}{15}y_1 + \frac{2}{7}y_2 + 0z &= 12 \\
0x_1 + 0x_2 + \frac{1}{7}y_1 + \frac{15}{7}y_2 + 0z &= 750
\end{align*}
\]

\[
\begin{array}{cccccc|c}
 x_1 & x_2 & y_1 & y_2 & z & RHS & Ratio \\
 0 & 1 & 1/14 & -1/7 & 0 & 15 & 15 \\
 1 & 0 & -2/15 & 1/7 & 0 & 12 & 12 \\
 0 & 0 & 5/7 & 15/7 & 1 & 750 & 750 \\
\end{array}
\]

Dependent variables \{x_1, x_2, z\}

Independent variables: $y_1 = y_2 = 0$

Extreme Point \((x_1, x_2) = (12, 15)\)

Value of Objective Function: $z = 750$