Euclid’s Elements
300 BCE
and Beyond
Euclid’s *Elements*

Book I

23 Definitions
5 Postulates
5 Common Notions
48 Propositions
Post. 1 – To draw a straight line from any point to any point.

Post. 2 – To produce a finite straight line continuously in a straight line.

Post. 3 - To describe a circle with any center and radius.

Post. 4 - That all right angles equal one another.
Post. 5 - That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
C.N. 1 - Things which equal the same thing also equal one another.

C.N. 2 - If equals are added to equals, then the wholes are equal.

C.N. 3 - If equals are subtracted from equals, then the remainders are equal.

C.N. 4 - Things which coincide with one another equal one another.

C.N. 5 - The whole is greater than the part.
Prop I.1 - To construct an equilateral triangle on a given finite straight line.
Prop. I.2 - To place a straight line equal to a given straight line with one end at a given point.
Prop. I.3 - To cut off from the greater of two given unequal straight lines a straight line equal to the less.
Prop. 1.4 – (a.k.a. SAS) If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proof uses “superposition” of triangle ABC onto triangle DEF where $AB = DE$, $AC = DF$ and the angles at $A$ and $D$ are equal.
Prop I.5 (Pons Asinorum) - In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.
Prop. 1.6 – (Converse of 1.5) If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.
Prop I.16 - In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

\[ \angle BAC = \angle ACF < \angle ACD \]
Prop. I-17 - In any triangle the sum of any two angles is less than two right angles.

It can also be shown that the sum of the angles in a triangle cannot exceed two right angles = 180 degrees. However, this and the previous Prop I.16 is not true in Elliptic Geometry.
Neutral Geometry: Any Propositions in Euclid’s Element *before* I.29

**Prop I.29** - A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.
Prop I.29

Alternate angles equal: $\angle 1 = \angle 2$

Exterior angle equal to interior & opposite interior angle: $\angle 2 = \angle 3$

Sum of interior angles on same side equal to two right angles: $\angle 1 + \angle 4 = 2$ right angles
Prop. I.32 - In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Proposition I.32 which uses Euclid’s 5th postulate states that the sum of the angles in a triangle is 180 degrees.
Note: The four relations for proving congruent triangle all come before Prop 1.29 meaning all are independent on the 5th Axiom

Prop 1.4    SAS
Prop 1.8    SSS
Prop 1.26   ASA and AAS
Prop I.47 – (Pythagorean Theorem) In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.
Prop I.48 – (converse of I.47) - If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.
Non-Euclidean Geometries

Playfair’s Postulate – Given a line \( l \) and a point \( P \) not on \( l \), there exists one and only one line \( m \) in the plane of \( P \) and \( l \) which is parallel to \( l \)

Elliptic Geometry: ... there exist no lines in the plane of \( P \) and \( l \) which is parallel to \( l \)

Hyperbolic Geometry: ... there exists more than one line in the parallel of \( P \) and \( l \) which is parallel to \( l \)
Saccheri’s Quadrilateral

Sides AD and BC are both perpendicular to AB and equal in length. Triangles ADB and BCA are congruent by SAS. It can be shown that angle ADC equals angle BCD and that side DC is parallel to side AB. Saccheri next assumed that angles ADC and BCD were greater than 90 degrees then less than 90 degrees. This led to contradictions (?)
Using I.16 and I.17 is can be shown that the angle sum of a triangle cannot exceed two right angles (180°).

Therefore the summit angles of the Saccheri quadrilateral cannot be greater than right angles (elliptic case).

Assuming the summit angles were less than right angles lead to a strange geometry (hyperbolic case).
In Hyperbolic Geometry the sum of the angles of a triangle is less than 180 degrees.

In Elliptic Geometry the sum of the angles of a triangle is greater than 180 degrees. In addition, lines are not infinite in length (see Post. 2)
Result: If the sum of the angles of a triangle is less than 2 right angles, AAA is a congruence relation.

Assume angles 1, 2, 3 equal angles 4, 5, 6 respectively but DE > AB. Show angle sum of triangle GHF = 180° !!!