Geometric Series: In class we showed how to find the sum of a converging geometric series by using the “shift technique” to find a closed form expression for \( s_n \), the \( n \)th partial sum (i.e. the sum of the first \( n \) terms) of the geometric series, then evaluating the limit \( \lim_{n \to \infty} s_n \) for the sequence of partial sums.

Example: What is \( \sum_{k=0}^{\infty} \left( \frac{1}{7} \right)^k = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots \)?

The shift technique:

1. Write out the \( n \)th partial sum (a finite sum)
   \[
   s_n = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots + \frac{1}{7^n}
   \]

2. Recall that for a geometric series there is a value \( r \) such that each successive term is \( r \) times the previous term. In this example \( r = \frac{1}{7} \). Multiply the \( n \)th partial sum by \( r \) (this causes the shift).
   \[
   \frac{1}{7} s_n = \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots + \frac{1}{7^{n+1}}
   \]

3. Subtract \( r \cdot s_n \) from \( s_n \). You will get \( s_n - r \cdot s_n = s_n (1 - r) \) on the left and a whole lot of cancellation of terms on the right!

   \[
   s_n - \frac{1}{7} s_n = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots + \frac{1}{7^n}
   \]

   \[
   \frac{1}{7} s_n = \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots + \frac{1}{7^n} + \frac{1}{7^{n+1}}
   \]

   \[
   s_n - \frac{1}{7} s_n = 1 - \frac{1}{7^{n+1}}
   \]
4. Now solve for a closed form for \( s_n \)

\[
s_n \left( 1 - \frac{1}{7} \right) = 1 - \frac{1}{7^{n+1}} \text{ or } s_n = \frac{7}{6} \left( 1 - \frac{1}{7^{n+1}} \right)
\]

5. Taking the limit of the closed form for \( s_n \) is easy yielding the value of the converging infinite series

\[
s_n = \frac{7}{6} \left( 1 - \frac{1}{7^{n+1}} \right) \to \frac{7}{6} \text{ as } n \to \infty \text{ hence } \sum_{k=0}^{\infty} \left( \frac{1}{7} \right)^k = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \ldots = \frac{7}{6}
\]

Do the Following: Using the above technique and showing all the work (demonstrating you understand what you are doing) evaluate the following converging geometric series using the shift technique to obtain the \( n \)th partial sum and finding the limit for the sequence of partial sums. Be sure to correctly determine \( r \) for each example. Note below that both 0 based and 1 based indexing are used.

#1 Find the sum \( \sum_{k=1}^{\infty} \left( \frac{1}{4} \right)^k = \frac{1}{4} + \left( \frac{1}{4} \right)^2 + \left( \frac{1}{4} \right)^3 + \ldots + \left( \frac{1}{4} \right)^k + \ldots \)

#2 Find the sum \( \sum_{k=1}^{\infty} \left( \frac{4}{17} \right)^k = \frac{4}{17} + \left( \frac{4}{17} \right)^2 + \left( \frac{4}{17} \right)^3 + \ldots + \left( \frac{4}{17} \right)^k + \ldots \)

#3 Find the sum \( \sum_{k=1}^{\infty} 2 \left( \frac{1}{9} \right)^k = 2 \left( \frac{1}{9} \right) + 2 \left( \frac{1}{9} \right)^2 + 2 \left( \frac{1}{9} \right)^3 + \ldots + 2 \left( \frac{1}{9} \right)^k + \ldots \)

#4 Find the sum \( \sum_{k=1}^{\infty} \left( \frac{-1}{6} \right)^k = \frac{-1}{6} + \left( \frac{-1}{6} \right)^2 + \left( \frac{-1}{6} \right)^3 + \ldots + \left( \frac{-1}{6} \right)^k + \ldots \) Note that the signs of this geometric series alternate!

#5 Find the sum \( \sum_{k=0}^{\infty} \left( \frac{2}{3} \right)^k = 1 + \left( \frac{2}{3} \right)^2 + \left( \frac{2}{3} \right)^3 + \ldots + \left( \frac{2}{3} \right)^k + \ldots \)
Every repeating decimal expansion can be expressed as a geometric series. For example

\[ 0.333\overline{3} = \sum_{k=1}^{\infty} 3 \left( \frac{1}{10} \right)^k = 3 \left( \frac{1}{10} \right) + 3 \left( \frac{1}{10} \right)^2 + \ldots + 3 \left( \frac{1}{10} \right)^k + \ldots \]

\[ 0.131313\overline{13} = \sum_{k=1}^{\infty} 13 \left( \frac{1}{100} \right)^k = 13 \left( \frac{1}{100} \right) + 13 \left( \frac{1}{100} \right)^2 + \ldots + 13 \left( \frac{1}{100} \right)^k + \ldots \]

**#6** Express the repeating decimal expansion \(0.111\overline{1}\) as a geometric series and use the techniques above to determine what the geometric series sums to. Note that your answer will be a fraction, a verification of the fact that every repeating decimal expansion represents a rational number.

**Instructions:** Hand in this cover sheet with your solution write ups. Your write ups should be clear and well organized; I want to see your work using the format given by the example for \(\sum_{k=0}^{\infty} \frac{1}{7^k}\). You are allowed to work together but you must do your own write ups (don’t copy someone else’s work).

**Extra Credit Challenge:** Every *eventually* repeating decimal (e.g. \(0.1\overline{666}\)) can be expressed as a finite sum plus a geometric series. Can you decompose \(0.1\overline{666}\) into a finite sum plus a geometric series then apply the techniques used about to find the sum of the geometric series to find what rational number \(0.1\overline{666}\) equals?