Polar Form for Complex Numbers

De Moivre’s Formula

1. **Complex Numbers:** Recall that a complex number $z$ is in the form $z = a + bi$ where $a$ and $b$ are real numbers and $i = \sqrt{-1}$; that is $i^2 = -1$.

2. **Arithmetic of Complex Numbers**
   
a. Addition and subtraction is component-wise:
   $$ (2 + 3i) \pm (5 - 7i) = (2 \pm 5) + (3 \pm 7)i $$
   
b. Use FOIL to multiply:
   $$ (2 + 3i)(5 - 7i) = 2 \cdot 5 + 2 \cdot (-7)i + 3 \cdot 5i + 3 \cdot (-7)i^2 = 31 + i $$
   
c. The complex conjugate of a complex number is obtained by negating the imaginary part. This is denoted by place a bar over the number: $(2+3i) = (2-3i)$. Any number multiplied by its conjugate yields a positive real (unless that number is zero).
   $$ (2+3i)(2-3i) = 2 \cdot 2 + 2 \cdot 3i - 2 \cdot 3i - 3 \cdot 3i^2 = 4 + 9 = 13 $$
   
   In general $(a + bi)(a - bi) = a^2 + b^2$. We refer to the square root $\sqrt{a^2 + b^2} = \sqrt{(a + bi)\cdot(a - bi)}$ as the magnitude (or modulus) of $a + bi$, often denoted using absolute values signs; i.e. $|a + bi| = \sqrt{a^2 + b^2}$. The magnitude of $a + bi$ equals 0 if and only if $a + bi = 0$

d. Division is done by multiplying numerator and denominator by the complex conjugate of the denominator:
   $$ \frac{2 + 3i}{5 - 7i} = \frac{(2 + 3i)(5 + 7i)}{(5 - 7i)(5 + 7i)} = \frac{-11 + 29i}{74} = -\frac{11}{74} + \frac{29}{74}i $$
3. **Geometry of Complex Numbers: The Complex Plane**

As the real number line is used to represent real numbers, the complex plane is used to represent complex numbers where the imaginary axis is erected at right angles to the real axis. There is a natural *one to one correspondence* between points in the complex plane and complex numbers of the form \( a + bi \) where \( a \) is the distance along the horizontal real axis and \( b \) is the distance along the vertical imaginary axis.

Since magnitude \( |a + bi| \) equals \( \sqrt{a^2 + b^2} \), by the Pythagorean distance formula, the magnitude of \( a + bi \), \( |a + bi| \) is the *distance* from the origin to the point \( a + bi \). Note that unlike real numbers, complex numbers are no longer ordered; i.e. given two complex numbers \( a + bi \) and \( c + di \) you can’t say either \( a + bi \leq c + di \) or \( c + di \leq a + bi \). Total ordering is the only property that complex number lack.\(^1\)

\[ a+bi \]

\[ |a+bi| \]

---

\(^1\) With apologies to Robert Heinlein’s *The Moon is a Harsh Mistress* I refer to this as the TANSTAAFL effect – “There Ain’t No Such Thing As A Free Lunch”. What we gain in algebraic closure we lose in total ordering.
4. Geometry of Addition & Subtraction – Parallelogram Rules

There is a simple geometric description of addition and subtraction. If you treat a complex number $a + bi$ like a vector (or arrow) whose tail is at the origin and whose head is at the point $a + bi$ (so the magnitude is the length of arrow), then the addition of two complex numbers can be seen as placing the tail of one vector at the head of the other with the resulting arrow being the sum.

For example $(3+2i) + (1+3i) = 4+5i$

As you can see the two arrows form two sides of a parallelogram – hence the parallelogram laws. To subtract reverse the subtrahend (subtractor) arrow and place its head (old tail) at the tail of the minuend (quantity subtracted from); i.e. $(1+3i) - (3+2i) = -2 + i$. The resulting arrow is parallel to the other diagonal of the parallelogram.
5. **The Polar Form for Complex Numbers:** 

\[ z = a + bi = re^{\theta} = r(\cos \theta + i \sin \theta) = r \cdot cis(\theta) \]

If \( \theta \) is the angle between the ray from the origin to the point \( z = a + bi \) and the positive real axis, then right triangle trigonometry shows that \( a = r \cos(\theta) \) and \( b = r \sin(\theta) \). Putting this together obtains

\[ a + bi = r \cdot \cos \theta + r \cdot \sin \theta \cdot i \]

where \( r = |z| = |a + bi| = \sqrt{a^2 + b^2} \) is the magnitude or modulus of \( z \) and \( \theta \) is called the angle or argument of \( z \). \( \theta \) is the angle whose tangent is \( \frac{b}{a} \) or in other words \( \theta = \tan^{-1} \left( \frac{b}{a} \right) \).

Thus \( z \) can be expressed as

\[ z = a + bi = re^{\theta} = r(\cos \theta + i \sin \theta) = r \cdot cis(\theta) \]

Using our knowledge of sine and cosine for certain angles we can convert between rectangular and polar representations. For example

\[ 2 + 2i = \sqrt{8} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \] where the argument of \( 2 + 2i \) is \( \frac{\pi}{4} \) and the magnitude is \( 2\sqrt{2} \)

\[ 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) = 1 + \sqrt{3}i \] where the argument of \( 1 + \sqrt{3}i \) is \( \frac{\pi}{3} \) and the magnitude is \( 2 \)

Watch what happens when we multiply
6. **Multiplication and Division of Complex Numbers: Rotations and Stretches**

If we multiply \(1 + \sqrt{3}i\) by \(i\) we obtain \((1 + \sqrt{3}i) \times i = (-\sqrt{3} + i)\) whose magnitude is 2 (check this for yourself). If we graph the product we obtain a right triangle whose sides are 1, \(\sqrt{3}\), 2.

![Complex Numbers Graph](image)

Observe that the argument of the product \(-\sqrt{3} + i\ (\frac{5\pi}{6})\) is the sum of the arguments the two factors \(\frac{\pi}{2}\) and \(\frac{\pi}{3}\). The magnitude is the product of the magnitudes of the two factors. Thus the effect of multiplication is a rotation (summing the arguments of the factors) plus a stretch (the magnitude of the product is the product of the two magnitudes). In other words – when you multiply two complex numbers you multiply their magnitudes and add their angles! Thus multiplication is a stretch and a rotation! In a similar way with division when you divide you divide by the magnitude of the denominator and subtract angles.

**Multiplication (and division) in polar form:** Let \(z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)\) and let \(z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)\). Then

\[
\begin{align*}
z_1 \times z_2 &= r_1 (\cos \theta_1 + i \sin \theta_1) \times r_2 (\cos \theta_2 + i \sin \theta_2) =
\end{align*}
\]

\[
\begin{align*}
r_1r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1)
\end{align*}
\]

Note the use of the sum formulas for sine and cosine$^2$

---

$^2$ \(\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)\) and \(\sin(A + B) = \cos(A) \sin(B) + \sin(A) \cos(B)\)
In a similar way with division when you divide you divide by the magnitude of the denominator and subtract angles.

\[
\frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \left( \cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2) \right)
\]

7. Polar Notation

It can be shown that \(e^{i\theta} = \cos \theta + i \sin \theta\), an interesting combination of the trig functions sine and cosine and the exponential function. Thus on the TI-83 & TI 84 calculators complex values in polar mode are given by \(re^{i\theta}\). For example \(2 + 2i = \sqrt{8}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = \sqrt{8}e^{i\frac{\pi}{4}}\).

Aside: Starting with \(e^{i\theta} = \cos \theta + i \sin \theta\), if we let \(\theta\) equal \(\pi\) and do a little algebra we obtain Euler’s formula \(e^\pi + 1 = 0\) which is called the most beautiful equation in mathematics since it combines \(e\), the natural exponent, \(i\), the square root of minus 1, \(\pi\), the ratio of the circumference of a circle to its diameter, 1, the multiplicative identity and 0, the additive identity.

8. Powers and Roots: De Moivre’s Theorem: \((\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)\)

When we square \(z = r(\cos \theta + i \sin \theta)\) we obtain \(z^2 = r^2 (\cos 2\theta + i \sin 2\theta)\). The same if true for cubic (angle triples) and fourth powers (angle goes up by a factor of 4) and in general for any positive integer \(n\) \(z^n = r^n (\cos n\theta + i \sin n\theta)\).

If \(|z| = 1\) then we have DeMoivre’s Theorem: \((\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\)

Finding roots now is easy. \(\sqrt[n]{z} = r^{1/n} \left( \cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right)\); that is, we divide the angle by \(n\). In general for \(n\)th roots \((n\) an integer) \(\sqrt[n]{z} = \sqrt[n]{r} \left( \cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right)\).

**Example** \(\sqrt{i} = \sqrt[4]{\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\). Check it out by squaring the result to obtain \(i\).

**Note:** Since \(i = \cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}\) then \(\sqrt{i} = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\). Remember that each number has two square roots!