Math 131- Chapter 3.4 – page 256 Exercise 46

a. \[ h(p) = \frac{p^3}{p^3 + (1 - p)^3} \] compute \( h'(p) \) and \( h''(p) \)

Given \[ h(p) = \frac{p^3}{p^3 + (1 - p)^3} = \frac{p^3}{p^3 + 1 - 3p + 3p^2 - p^3} = \frac{p^3}{1 - 3p + 3p^2} \]

it’s easier to expand the denominator and cancel the \( p^3 \) term

Since \( p \) is a proportion, the domain is restricted to \([0, 1]\). If we look for zeroes in the denominator which is a quadratic, then using the quadratic formula \[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \] the value under the radical is \((-3)^2 - 4 \cdot 3 \cdot 1 = -3\) so there are no real zeros. Thus \(1 - 3p + 3p^2\) is always positive and so the function is actually defined everywhere.

Calculating the derivative using the quotient rule we obtain

\[
h'(p) = \frac{(1 - 3p + 3p^2)3p^2 - p^3(-3 + 6p)}{(1 - 3p + 3p^2)^2} =
\]

\[
\frac{3p^2 - 9p^3 + 9p^4 + 3p^2 - 6p^4}{(1 - 3p + 3p^2)^2} =
\]

\[
\frac{3p^4 - 6p^3 + 3p^2}{(1 - 3p + 3p^2)^2}
\]

This is the derivative in factored form. Critical numbers are 0 and 1. Using an arrow diagram the derivative is always positive so the function is increasing everywhere. There are no relative extrema since the function is always increasing. Observe the points (0,0) and (1,1) are on the graph.
The second derivative is a bit harder to obtain. Again using the quotient rule

\[ h''(p) = \frac{(1-3p^2)^2 \left( 3p^2 \cdot 2(1-p) \cdot (-1) + (1-p)^2 \cdot 6p \right) - 3p^2 (1-p)^2 \left( 2 \cdot (1-3p^2) \cdot (-3+6p) \right)}{(1-3p^2)^4} \]

The numerator term is ugly with two terms but the trick is to factor out as many common terms as possible.

\[
\begin{align*}
 h''(p) &= \frac{(1-3p^2)^2 \left( 6p \cdot (1-p) \cdot (-p + 1-p) \right) + 18p^2 \cdot (1-p)^2 \left( 1-3p^2 \right) \cdot (1-2p)}{(1-3p^2)^4} \\
 &= \frac{6p \cdot (1-3p^2) \cdot (1-p) \cdot (1-2p) \cdot \left( (1-3p^2) + 3p(1-p) \right)}{(1-3p^2)^4} \\
 &= \frac{6p \cdot (1-3p^2) \cdot (1-p) \cdot (1-2p) \cdot (1-3p^2 + 3p - 3p^2)}{(1-3p^2)^4} \\
 &= \frac{6p \cdot (1-3p^2) \cdot (1-p) \cdot (1-2p)}{(1-3p^2)^4}
\end{align*}
\]

Again we obtain a fairly nice factored form of the 2\textsuperscript{nd} derivative which has zeros at 0, \( \frac{1}{2} \) and 1. Checking the sign of the 2\textsuperscript{nd} derivative, the function is concave down on \((-\infty, 0)\), concave up on \((0, \frac{1}{2})\), concave down on \((\frac{1}{2}, 1)\) and concave up on \((1, +\infty)\)

The inflection point \(( \frac{1}{2}, \frac{1}{2} )\) is on the graph

b. To graph the function on the interval, plot the points \((0,0)\), \((\frac{1}{2}, \frac{1}{2})\) and \((1,1)\). The graph is increasing concave up on \([0, \frac{1}{2})\) and increasing concave down on \((\frac{1}{2}, 1)\)

c. \( h(0.61) = 0.7928082432 \) which is about 79\%