Mean Value Theorem: If \( f(x) \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\) then there exists a point \(c\) on \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

In other words – somewhere there is a derivative (slope of red line) equal to the average rate of change (slope of dotted line) over the interval or somewhere the instantaneous rate of changes equals the average rate of change.

Applications of the Mean Value Theorem:

Example: \( f(x) = 4 - x^2 \) is differentiable on the closed interval \([-1, 3]\). Find the point \(c\) on this interval where \( f'(c) = \frac{f(3) - f(-1)}{3 - (-1)} \) and compute the equation of the tangent line

Solution:

\[
\frac{f(3) - f(-1)}{3 - (-1)} = \frac{4 - 3^2 - (4 - (-1)^2)}{4} = \frac{-5 - 3}{4} = -2
\]

\( f'(x) = -2x \)

Solving \(-2x = -2\) yields \( x = 1 \)

\( f(1) = 3 \)

Equation of tangent line is \( y = -2(x - 1) + 3 = -2x + 5 \)
Example: A person enters the state thruway at 1 PM. At 2:06 PM he exits the thruway from a point 75 miles away. He is immediately arrested for breaking the 65 MPH speed limit although this “act” of speeding was never directly observed. Why?

Answer: Compute his average speed (75 miles in 1.1 hours) and apply the MVT

Example: The Mean Value Theorem will be used in the proof of the 1st Fundamental Theorem of Calculus.

Why is the MVT true? Start with the Max-Min Theorem

Max-Min Theorem: If \( f(x) \) is differentiable on the open interval (a,b) and if \( f(x) \) has a local maximum (minimum) at \( c \) for \( a < c < b \) then \( f'(c) = 0 \).

Informal Proof: Assume \( f(x) \) has a local maximum at \( c \) meaning that for all \( x \) sufficiently close to \( c \), \( f(x) \leq f(c) \). Suppose \( x \) is some point to the left of \( c \) (i.e. \( x < c \)) and consider the slope of the secant line thru \( x \) and \( c \) which we know approximates \( f'(c) \), the derivative at \( c \).

\[ \begin{array}{c}
\text{Since } x < c \text{ and } f(x) \leq f(c) \text{ the slope of the secant will always be } \geq 0 \text{ (see diagram). Since the slope of the secant gets closer and closer to } f'(c) \text{ as } x \text{ get closer and closer to } c \text{ and since this slope is always positive, we conclude that } f'(c) \geq 0. \\
\end{array} \]

A similar argument (not shown) where \( x \) is some point to the right of \( c \) (i.e. \( c < x \)) determines a secant line whose slope is negative. Since the slope of this second secant gets closer and closer to \( f'(c) \) as \( x \) get closer and closer to \( c \) and since this slope is always negative, we conclude that \( f'(c) \leq 0 \).

Combining these two conditions (\( f'(c) \geq 0 \) and \( f'(c) \leq 0 \)) we conclude \( f'(c) = 0 \! \text{ QED} \)

Okay where does that leave us? Rolles’ Theorem
**Rolle's Theorem:** If \( f(x) \) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\) and \( f(a) = f(b) \) then there is a point \( c \) on \((a, b)\) such that \( f'(c) = 0 \).

Informal Proof: If \( f(x) \) is constant on \([a, b]\) then trivially \( f'(c) = 0 \) for all \( c \) on \([a, b]\); otherwise either \( f(x) > 0 \) somewhere on \((a, b)\) which means \( f(x) \) has a local maximum or \( f(x) < 0 \) somewhere on \((a, b)\) which means \( f(x) \) has a local minimum. In either case, by the **Max-Min Theorem** \( f'(c) = 0 \) for some \( c \).

In other words: any continuous differentiable function anchored at both end to 0 that “goes up” must turn around and “go down”.

Diagram of Rolle's Theorem

\[
\begin{align*}
\text{Applications of Rolle's Theorem: Proving the MVT} \\
\text{Proof of the **Mean Value Theorem**: The proof of the MVT used a very cleverly defined function} \\
g(x) \text{ which makes use of the function } f(x) \text{ and the difference quotient } \frac{f(b)-f(a)}{b-a} \text{ for the end} \\
\text{points of the interval. That is} \\
g(x) = \frac{f(b)-f(a)}{b-a} (x-a) - \left( f(x) - f(a) \right).
\end{align*}
\]

Being constructed from differentiable functions \((x-a)\) and \((f(x) - f(a))\) - note that \( f(a) \) is a constant - \( g(x) \) is differentiable. Moreover at the intervals endpoints \( g(a) = 0 = g(b) \) (check this out for yourself!)

Thus by **Rolle's Theorem** the derivative of \( g(x) \), \( g'(c) = \frac{f(b)-f(a)}{b-a} - f'(c) = 0 \) for some \( c \) on \((a, b)\) or solving for \( f'(c) \)

\[
f'(c) = \frac{f(b)-f(a)}{b-a}.
\]
Another Application of the MVT

**Result:** If \( f(x) \) is a differentiable function on some interval \( I \) such that \( f'(x) = 0 \) for all \( x \) on \( I \), then \( f(x) = c \), that is \( f(x) \) is a constant function.

Assume false, that is \( f(x) \) is not a constant function on interval \( I \). Then there are two points on \( I \), \( a \) and \( b \) such that \( f(a) \neq f(b) \). Hence by the MVT there is a point \( c \) on \( I \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a} \neq 0
\]

But this contradicts that fact that \( f'(x) = 0 \) for all \( x \) on \( I \). Therefore \( f(x) = c \) is constant on \( I \).

**Exercises**

1. \( f(x) = x^3 \) is differentiable on the interval \([0, 2]\). Using the MVT find the point \( c \) on the interval where \( f'(c) \) equals the average rate of change of the function \( f(x) \) over the interval \([0, 2]\).

2. Do the same for \( f(x) = x^2 + x + 1 \) on the interval \([-2, 4]\).

3. Using the MVT prove in general that for any quadratic \( f(x) = \alpha x^2 + \beta x + \gamma \) (note the use of Greek letters in place of \( a, b, \) and \( c \)) and for any interval \([x_1, x_2]\) the point \( c \) obtained from the MVT is always the midpoint of the interval \([x_1, x_2]\); that is show that \( c = \frac{x_1 + x_2}{2} \).