Integral Functions and the 2\textsuperscript{nd} Fundamental Theorem of Calculus

Recall that for a function \( f(x) \) which is continuous on a closed interval \([a,b]\), the definite integral \( \int_a^b f(x) \, dx \) which is the limit of a Riemann sum is the area under the curve. It’s a number.

Starting with the definite integral and we can turn it into a function by making the upper limit of integration the variable so that the integral function \( A(x) = \int_a^x f(t) \, dt \) returns the area under the function \( f(t) \) between a and (the variable) \( x \).

Note the change of notation for the integrand: \( f(t) \) instead of \( f(x) \). The function being integrated is still the same but so as not to confuse the variable used by the integrand with the variable \( x \) used by the integral function, the change the former to \( t \).

Aside: Think of the integral function \( A(x) = \int_a^x f(t) \, dt \) as a 3\textsuperscript{rd} kind of integral after the indefinite integral \( \int f(x) \, dx \) (anti-derivative) which is a (class of) function(s) and the definite integral \( \int_a^b f(x) \, dx \) (area under the curve) which is a number.

Example: \( A(x) = \int_0^x t \, dt \) is the area of a triangle whose side is length \( x \)

Since we know to find the area of a right triangle with side \( x \), we can evaluate the definite integral function \( A(x) = \int_0^x t \, dt = \frac{x^2}{2} \). If we formally evaluated the integral using the FTC, we’d obtain the same answer.
Example: If for the *same integrand* we make the lower limit of integration equal to 1 we obtain a different integral function: \( A(x) = \int_1^x t \, dt = \frac{x^2}{2} - \frac{1}{2} \) (Why?). This function is the area of a certain kind of trapezoid.

\[ \int_1^x x \, dt = \frac{x^2}{2} - \frac{1}{2} \]

The Integral Function: Given a function \( f(t) \) which is continuous on the interval \([a, b]\) containing \( x \), we define the integral function \( A(x) = \int_a^x f(t) \, dt \) as the function which returns the area under the curve \( f(t) \) between \( a \) and \( x \).

The left endpoint at \( a \) is fixed; the right endpoint at \( x \) is a variable; so it can vary (move). And as \( x \) moves the area under the curve of \( f \) changes. So \( A(x) \) is a function.

Example: If \( x = a \) what is \( A(a) \), the area under the curve when both endpoints are the same?

Answer: \( A(a) = 0 \)

Question: True or False: For \( x < a \) and \( f(t) \) continuous does \( \int_a^x f(t) \, dt = -\int_x^a f(t) \, dt \)?

Think definite integrals with fixed \( x \).
The 2\textsuperscript{nd} Fundamental Theorem of Calculus proves that \( A(x) = \int_a^x f(t) \, dt \) is a differentiable function whose derivative is \( f(x) \); that is \( \frac{d}{dx}(A(x)) = f(x) \).

**The 2\textsuperscript{nd} Fundamental Theorem of Calculus:** If \( f \) is a continuous function on an interval containing \( a \) then the definite integral function \( A(x) = \int_a^x f(t) \, dt \) defined as the area under the curve between \( a \) and \( x \) is the anti-derivative of \( f \); or to put it another way \( \frac{d}{dx}(A(x)) = f(x) \).

Proof \( \frac{d}{dx}(A(x)) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \int_x^{x+h} f(t) \, dt = \lim_{h \to 0} \frac{f(x^*) \cdot h}{h} = \lim_{h \to 0} f(x^*) = f(x) \)

We start off with the definition of derivative. Since we’re looking at the difference of two areas \( A(x+h) \) and \( A(x) \) we can express this as the definite integral \( \int_x^{x+h} f(t) \, dt \). This latter is the area of a small rectangle-like area under the curve. It can be shown that there is some point \( x^* \) between \( x \) and \( x+h \) such that the area of the rectangle of height \( f(x^*) \) and width \( h \) which is \( f(x^*) \cdot h \) equals this area (see diagram).

Since \( f \) is continuous, as \( h \) gets smaller, \( f(x^*) \) approaches \( f(x) \) since \( x^* \) is between \( x \) and \( x+h \).

Thus the derivative of \( A(x) \) equals \( f(x) \); that is \( \frac{dA}{dx} = f(x) \)

\[
\text{QED}
\]
In the case that \( f(x) \) has an explicitly known anti-derivative \( F(x) \) we have

\[
A(x) = \int_{a}^{x} f(t) \, dt = F(x) - F(a)
\]

For example

\[
\int_{1}^{3} t^2 \, dt = \frac{x^3}{3} - \frac{1}{3}
\]

**So why is this useful?**

Consider the following integrals

\[
\begin{align*}
\int e^{kx} \, dx &= \frac{1}{k} e^{kx} + C \\
\int x \cdot e^{x^2} \, dx &= \frac{1}{2} e^{x^2} + C \\
\int x \cdot e^{x^2} \, dx &= x \cdot e^{x^2} - e^{x} + C \\
\int e^{-x^2} \, dx &= \int e^{-x^2} \, dx + C
\end{align*}
\]

The first three can be integrated using an elementary formula, u substitution and integral by parts respectively. Unfortunately there is no elementary function for the 4th integral (which is actually a useful integral). However we can define the anti-derivative functions as

\[
\int e^{-x^2} \, dx = \int_{a}^{x} e^{-t^2} \, dt + C.
\]

As it turns out there are a number of useful functions that can only be defined *as integral functions* which are all well-behaved since by the 2nd Fundamental Theorem of Calculus, they are all *differentiable* (and hence *continuous*).

Moreover using numerical technical of integration, values for integral functions can be computed to any degree of precision!