Math 171 – Discrete Mathematical Structures
Today’s Overview

Composition of Functions

1:1 functions
Onto functions
Cardinality

Finite Sets
Countably Infinite Sets
R is uncountable

Composition of Functions

If \( g: A \rightarrow B \) and \( f: B \rightarrow C \) are functions we define the composition of function \( f = g: A \rightarrow C \) as follows: For all \( x \in A \), \( f \circ g(x) = f(g(x)) \in C \).

Examples: If \( g(x) = x + 1 \) and \( f(x) = x^2 \), then
\[
(f \circ g)(x) = f(g(x)) = (x+1)^2 \quad \text{and} \quad g \circ f(x) = g(f(x)) = x^2 + 1
\]

Arrow Diagram Examples

Composition of One-to-One and Onto Functions

Result: If \( g: A \rightarrow B \) and \( f: B \rightarrow C \) are one-to-one functions then the composition \( f \circ g: A \rightarrow C \) is one-to-one

Pf: Show if \( f(g(x_1)) = f(g(x_2)) \) then \( x_1 = x_2 \).
Since \( f \) is one-to-one then by definition \( g(x_1) = g(x_2) \).
Since \( g \) is one-to-one then again by definition \( x_1 = x_2 \).

Composition of Onto Functions

Result: If \( g: A \rightarrow B \) and \( f: B \rightarrow C \) are onto then their composition \( f \circ g \) is onto.

Pf: To show \( f \circ g \) is onto show for any \( z \in C \) there is an \( x \in A \) such that \( f(g(x)) = z \). Since \( f \) is onto there is a \( y \in B \) such that \( f(y) = z \). Likewise since \( g \) is onto, for \( y \in B \) there is an \( x \in A \) such that \( g(x) = y \). Thus \( f(y) = f(g(x)) = z \) and the result follows.

Galileo’s Paradox: “There are as many squares as there are numbers because they are just as numerous as their roots.”
However \( \mathbb{Z}^+ \) is properly contained in the set of all squares !!!!

Cardinality: Two sets have the same cardinality if and only if there is a one-to-one correspondence between them
For sets \( A, B \) and \( C \), cardinality is
reflexive
symmetric
transitive.

Notation: For a set \( S \) let \( |S| \) denote the cardinality of \( S \)

Finite Sets
A set \( S \) is finite (or has finite cardinality) if and only if there is a 1:1 correspondence between a finite sequence of positive integers \( S_n = \{1, 2, 3, \ldots, n\} \) and \( S \). In this case we say \( |S| = n \)

Countably Infinite Sets
A set \( S \) is countably infinite if and only if there is a one-to-one correspondence between \( S \) and \( \mathbb{Z}^+ \).
The following sets are countably infinite: \( 2\mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}^+, \mathbb{Q} \)
Proving Cardinality of Two Sets are Equal

Show \( f: \mathbb{Z}^+ \rightarrow 2\mathbb{Z}^+ \) defined by \( f(n) = 2n \) is 1:1 and onto

One-to-one: Let \( f(n) = f(m) \). Then \( 2n = 2m \) or \( n = m \).

Onto: Let \( n \in 2\mathbb{Z}^+ \). Then \( n = 2k \) for some \( k \in \mathbb{Z}^+ \). Thus \( f(k) = 2k = n \)

Show \( \mathbb{Z} \) is countably infinite; Define \( f: \mathbb{Z}^+ \) to \( \mathbb{Z} \) as follows

\[
    f(n) = \begin{cases} 
        \frac{n}{2} & \text{if } n \text{ is even} \\
        -\frac{n-1}{2} & \text{if } n \text{ is odd}
    \end{cases}
\]

Countable Sets

A set \( S \) is countable if and only if it is finite or countably infinite.

A finite set \( S \) cannot be put into a 1:1 correspondence with a proper subset of itself; an infinite set \( S \) can be put into a 1:1 correspondence a proper subset of itself

Uncountable Sets

\( \mathbb{R} \), the set of all real numbers is uncountable; that is there is no one-to-one correspondence between \( \mathbb{R} \) and \( \mathbb{Z}^+ \). In particular there is no 1:1 correspondence between \( (0,1) \) and \( \mathbb{Z}^+ \).

Aside: Prove \( 0.99999 = 1 \)

Written Homework #28 – Due F 11/15/13

Exercise Set 7.4 (page 459)

#2, #3, #10, #15