Cardano’s Solution of the Cubic

The State of Mathematics in the 16th Century – Intrigue & Deception: From Pacioli to Cardano
Cardano: His life and his mathematics
Solving a Depressed Cubic \( x^3 + mx = n \)
Converting a general cubic \( ax^3 + bx^2 + cx + d \) to the depressed form & putting it all together
Solving \( x^3 - 15x = 4 \) and dealing with complex numbers
Viete’s method for the Depressed Cubic
The Depressed Quadratic & The quadratic formula
To the Quartic and beyond
Ballistics: Aristotle to Tartaglia to Galileo to Newton

Cardano and the Solution of the Cubic

- State of Mathematics ca. 1500 CE
- Luca Pacioli (1445-1509) - *Summa de Arithmetica*
- Scipione del Ferro (1465 – 1526) the depressed cubic
- Antonio Fior (ca. 1506 - ?) – student of del Ferro
- Niccolo Fontana (Tartaglia) (1499-1557) – also solved depressed cubic
- Gerolano Cardano (1501-1576)- solved general cubic
- Ludovico Ferrari (1522-1563) – student of Cardano
Gerolamo Cardano of Milan (1501-1576)

- His birth? His health!
- Cardano the Physician (in Sacco – 1526 - 1532)
- His marriage (1531) to Lucia Bandarini (woman in white)
- His return to Milan (1532) – Triumph & Tragedy
  death of wife (1546) & trouble with children (1557, 1560)
- March 25, 1539 – Tartaglia reveals his secret to Cardano
- 1545 Ars Magna
- 1570 Heresy (?) to Pension from the Pope!

Cardano and the Solution of the Cubic

Theorem: Rule to solve “cube and first power equal to number”

\[ x^3 + mx = n \]

“Cube one third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same . . . Then, subtracting the cube root of the first from the cube root of the second, the remainder which is left is the value of x”

Cardano started by decomposing a large cube of side t ...
Solving the Depressed Cubic: Slicing a Cube

Start with the decomposition of a cube with side $t$

\[
t^3 = u^3 + (t-u)^3 + 2tu(t-u) + u^2(t-u) + u(t-u)^2
\]

\[
u^2(t-u) + u(t-u)^2 = u(t-u)(u+t-u) = ut(t-u)
\]

\[
t^3 = u^3 + (t-u)^3 + 3ut(t-u)
\]

\[
(t-u)^3 + 3tu(t-u) = t^3 - u^3
\]

\[
x^3 + mx = n
\]

Solving the Depressed Cubic: A New System

\[3tu = m\]

\[t^3 - u^3 = n\]

Two equations in two unknowns

non-linear!

\[u = \frac{m}{3t}\]

\[t^3 - \frac{m^3}{27t^3} = n\]

\[\Rightarrow t^6 - mt^3 - \frac{m^3}{27} = 0\]

A quadratic in $t^3$

\[t^3 = \frac{n \pm \sqrt{n^2 + \frac{4m^3}{27}}}{2}\]

\[= \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}\]

Using Quadratic formula

\[t = \sqrt[3]{\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}\]
The Depressed Cubic: Putting It all Together

\[
t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}
\]

\[
u^3 = t^3 - n \Rightarrow u^3 = \frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}} - n
\]

\[
u = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}
\]

\[
x = t - u = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}
\]

What Cardano Found: Converting to the Depressed Form

Replace \( x = \left( y - \frac{b}{3a} \right) \)

\[a\left( y - \frac{b}{3a} \right)^3 + b\left( y - \frac{b}{3a} \right)^2 + c\left( y - \frac{b}{3a} \right) + d \]

eliminate the \( y^2 \) term

\[a\left( y^3 - \frac{3b}{2a} y + \frac{3b^2}{27a^2} - \frac{b^3}{27a^3} \right) +
\]

\[b\left( y^2 - \frac{2b}{3a} y + \frac{b^2}{9a^2} \right) + c\left( y - \frac{b}{3a} \right) + d \]

\[y^3 + \left( \frac{3b^2}{9a^2} - \frac{2b^2}{3a^2} + \frac{c}{a} \right) y = \left( \frac{b^3}{27a^3} + \frac{cb^2}{3a^2} \right) \]
Putting It All together

Given: \( ax^3 + bx^2 + cx + d = 0 \)

1. Substitute \( y - \frac{b}{3a} \) for \( x \). Reduce to a reduced cubic solving for \( m \) and \( n \) in terms of \( a, b, c, \) and \( d \).

2. Solve the reduced cubic
\[
y = \sqrt[3]{\frac{n}{2}} + \sqrt[3]{\frac{n^2}{4} + \frac{m^3}{27}} - \sqrt[3]{\frac{n}{2}} + \sqrt[3]{\frac{n^2}{4} + \frac{m^3}{27}}
\]

3. Substitute back: \( x = y + \frac{b}{3a} \)

Example

\[2x^3 - 30x^2 + 162x - 350 = 0\]

\[2 \left( y + 5 \right)^3 - 30 \left( y + 5 \right)^2 + 162 \left( y + 5 \right) - 350 = 0\]

\[2y^3 + 12y - 40 = 0\]

\[y^3 + 6y = 20\]

\[y = \sqrt[3]{\frac{20}{2}} + \sqrt[3]{\frac{20^2}{4} + \frac{6^3}{27}} - \sqrt[3]{\frac{20}{2}} + \sqrt[3]{\frac{20^2}{4} + \frac{6^3}{27}}\]

\[y = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}} = 2\]

\[x = 7\]
Solving \( x^3 - 15x = 4 \)

\[ x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{2 + \sqrt{-121}} \]

Rafel Bombelli (ca. 1526 – 1573) *Algebra*: Observe:

\[ (2 + \sqrt{-1})^3 = (2 + \sqrt{-1})(3 + 4\sqrt{-1}) = 2 + 11\sqrt{-1} = 2 + \sqrt{-121} \]

\[ (-2 + \sqrt{-1})^3 = (-2 + \sqrt{-1})(3 - 4\sqrt{-1}) = -2 + 11\sqrt{-1} = -2 + \sqrt{-121} \]

So \( 2 + \sqrt{-1} = \sqrt[3]{2 + \sqrt{-121}} \) and \( 2 - \sqrt{-1} = \sqrt[3]{2 - \sqrt{-121}} \)

Therefore \( x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{2 - \sqrt{-121}} = 4 \)

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Complex Numbers: Enter Euler (18th Century)

Start with \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

Also:

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \]

Therefore \( e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos(x) + i\sin(x) \)
Euler & DeMoivre

Euler’s Formula: $e^{i\theta} = \cos(\theta) + i\cdot\sin(\theta)$

Note $|\cos(\theta) + i\cdot\sin(\theta)| = 1$

DeMoivre’s Formula:

$$(\cos(\theta) + i\cdot\sin(\theta))^n = \cos(n\theta) + i\cdot\sin(n\theta)$$

Proof by induction on $n$ using the trig identities

$$\cos(2a) = \cos^2(a) - \sin^2(a)$$
$$\sin(2a) = 2\sin(a)\cos(a)$$

DeMoivre & the $n$ th Roots of Unity

Since by DeMoivre

$$(\cos(\theta/n) + i\cdot\sin(\theta/n))^n = \cos(\theta) + i\cdot\sin(\theta)$$

it follows that

$$(\cos(\theta)+i\cdot\sin(\theta))^{1/n} = \cos(\theta/n)+i\cdot\sin(\theta/n)$$

Also

$$\cos((2k\pi+\theta)/n) + i\cdot\sin((2k\pi+\theta)/n))^n =$$

$$\cos(2k\pi+\theta) + i\cdot\sin(2k\pi+\theta) = \cos(\theta) + i\cdot\sin(\theta)$$

for $k = 0, 1, 2, ... n-1$

It follows that there are $n$ roots

$$\frac{(\cos(\theta)+i\cdot\sin(\theta))^{1/n} = \cos((2k\pi+\theta)/n)+i\cdot\sin((2k\pi+\theta)/n)}{\text{for } k = 0, 1, 2, ... n-1}$$
The Geometry of Complex Numbers: Polar form

Therefore: a+bi has a polar form: $r \cdot (\cos(\theta) + i \cdot \sin(\theta))$

Finding $\sqrt[3]{2+11i}$

Start with the polar form for complex numbers and DeMoivre’s formula $(\cos(\theta)+i \cdot \sin(\theta))^n = \cos(n\theta)+i \cdot \sin(n\theta)$

$$2+11i = \sqrt[3]{125} \left( \cos \theta + i \sin \theta \right) \quad \theta = \tan^{-1}\left( \frac{11}{2} \right)$$

$$\sqrt[3]{2+11i} = \sqrt[3]{125} \left( \cos \frac{\theta}{3} + i \sin \frac{\theta}{3} \right)$$

$$\tan^{-1}\left( \frac{11}{2} \right) = 1.390942827$$

$$\cos\left( \frac{\theta}{3} \right) = 0.894427191 \quad \sin\left( \frac{\theta}{3} \right) = 0.4472135955$$

$$\sqrt[3]{2+11i} = \sqrt[3]{5} \left( 0.894427191 + 0.4472135955i \right) = 2 + i$$
Dealing with $x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{2 - \sqrt{-121}}$

Using TI-84: $\sqrt[3]{2 + \sqrt{-121}} = 2 + i$

However $\sqrt[3]{2 - \sqrt{-121}} = 1.866025404 + 1.2320508808i$

To get Bombelli’s value (rotate) multiply by $
\cos(2\pi/3) + i \cdot \sin(2\pi/3)$

$\sqrt[3]{2 + \sqrt{-121}} \cdot (\cos \left(\frac{2\pi}{3}\right) + i \cdot \sin \left(\frac{2\pi}{3}\right)) = -2 + i$

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**The Depressed Quadratic → Quadratic Formula**

Given the quadratic $ax^2 + bx + c = 0$ substitute $y = x - \frac{b}{2a}$

$\left(y - \frac{b}{2a}\right)^2 + b \left(y - \frac{b}{2a}\right) + c = $

$ay^2 - by + \frac{b^2}{4a} + by - \frac{b^2}{2a} + c = $

$ay^2 + \frac{-b^2 + 4ac}{4a} = 0 \Rightarrow y^2 = \frac{b^2 - 4ac}{4a^2}$
Viete’s (1540 – 1603) Method for the Depressed Cubic

Start with $x^3 + mx = n$ and substitute $x = \frac{m}{3y} - y$ which yields

\[
\left( \frac{m}{3y} - y \right)^3 + m \left( \frac{m}{3y} - y \right) = n \Rightarrow \frac{m^3}{27y^3} - y^3 = n
\]

\[
\frac{m^3}{27y^3} - y^3 = n \Rightarrow y^6 + ny^3 - \frac{m^3}{27} = 0
\]

\[
\therefore y^3 = \frac{-n \pm \sqrt{n^2 - 4\left( \frac{m^3}{27} \right)}}{2} = \frac{-n \pm \sqrt{n^2 + \frac{m^3}{4}}}{2}
\]

so \[ y = \sqrt[3]{\frac{-n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \]

now solve for $x$: \[ x = \frac{m}{3y} - y \]

To the Quartic and Beyond ...

Cardano cited Ludovico Ferrari’s solution to the quartic
1. Convert to depressed quartic \[ y^4 + my^2 + ny = p \]
2. Reduce depressed quartic to cubic
3. Solve the cubic

So to solve the quintic equation...

Neils Abel (1802 – 1829) showed there is no “solution by radicals” for the general quintic equation
Ballistics

Ballistics: “the science of mechanics that deals with the flight, behavior and effects of projectiles”

Aristotle: Two types of motion
- Natural: tendency of all things to return to their proper place (i.e. a falling stone; rising smoke)
- Unnatural: requires force acting on object at all times (no concept of inertia)

A cannon ball flies on straight line until it reaches end of its flight when it drops

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Tartaglia & Ballistics

In 1531 asked “at what angle of elevation would a cannon achieve it’s greatest range?”

First to study trajectories and ballistics
Decided 45° angle would achieve longest shot
Invented “gunner’s quadrant”
Asserted that artillery cannot shoot in straight line; there is tendency for trajectory to bend
Drew up ballistic tables for cannons with range based on angles of elevation and charges (used until 17th century)
Experimented with gunpowder, weight and diameter of projectiles – tried to find ideal length of gun barrel
Tartaglia & Ballistics

In anguish over war’s destruction and overcome with moral dilemma of his work, he destroyed his writings

However, afterward an alliance of France with Ottoman Empire against Italy caused him to reconstruct work from *Gunpowder* by Jack Kelly

Galileo (1564 – 1642) & Ballistics

Revised study of ballistics

1. Assumed projectile encounters no resistance
2. Resolved trajectory into two components: force of gunpowder explosion and force of gravity
3. Concept of inertia
4. Acceleration (dv/dt) is force needed only to change velocity instead of maintaining velocity
5. Trajectory was parabolic curve.

Isaac Newton b. 12/25/1642
Ballistics

Solution to a 2nd Degree Differential Equation:

\[
\frac{d^2y}{dt^2} = -g + k \cdot \frac{dy}{dt}
\]

\[
\frac{dv}{dt} = -g + k \cdot v
\]