Mersenne Primes and Perfect Numbers – Write Up Carefully!

A. **Mersenne Primes**: A Mersenne prime is a prime of the form $2^p - 1$. For example, $7 = 2^3 - 1$ is a Mersenne prime.

1. Prove the following: If $2^p - 1$ is prime then $p$ is prime.

**Hint**: Prove the *contrapositive*; that is, prove if $p$ is *not* prime then $2^p - 1$ is *not* prime. Do this by factoring $2^p - 1$ for $p$ a composite. For example: $2^{15} - 1 = (2^3 - 1)(2^{12} + 2^9 + 2^6 + 2^3 + 1)$.

Multiply it out yourself; there is a pattern to this! Extend it to $2^{ab} - 1$. That is, if $(2^{ab} - 1) = (2^a - 1)(2^b + ... + 1)$ for $a, b > 1$, what should the exponents be for the 2nd factor?

2. Show by counter-example that the converse is *false*: That is, “if $p$ is prime then $2^p - 1$ is prime” is a *false* statement. Find a prime $p$ for which $2^p - 1$ is composite.

B. **Perfect Numbers**. Euclid proved that if $2^p - 1$ is a (Mersenne) prime, then $(2^p - 1) \times 2^{p-1}$ is a perfect number. For example, since $2^5 - 1 = 31$ is prime, $(2^5 - 1) \times 2^{5-1} = 31 \times 2^4 = 31 \times 16 = 496$ is perfect since $496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$. Note that we can also express this as

$$496 = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + (31 \times 2^0) + (31 \times 2^1) + (31 \times 2^2) + (31 \times 2^3)$$

This should provide some insight as to how to prove this in general.

Prove it by showing the following:

1. First using induction verify $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for *any* positive integer $n$. You will need this result later.

2. For $2^p - 1$ prime, show that all proper divisors of $(2^p - 1) \times 2^{p-1}$ are either powers of 2 (Type I) or are $(2^p - 1)$ times a power of 2 (Type II).

For example, with 496 the Type I divisors are 1, 2, 4, 8 and 16 while the Type II divisors are 31, 62, 124, and 248 (don’t count 496 itself).
3. Now sum all the proper divisors.

Using the induction result for A, the Type I divisors “sum” to \(2^p - 1\) (i.e. 
\[1 + 2 + 4 + 8 + 16 = 2^5 - 1\]) while Type II divisors sum to \((2^p - 1)(2^{p-1} - 1)\) (i.e. 
\[31 + 62 + 124 + 248 = 31(1 + 2 + 4 + 8) = (2^5 - 1)(2^4 - 1)\])

So the sum of all the divisors (Type I and Type II) is

\[(2^p - 1) + (2^p - 1)(2^{p-1} - 1) = (2^p - 1)(1 + 2^{p-1} - 1) = (2^p - 1)(2^{p-1})\]