Three Famous Problems in Classical Greek Mathematics

Background: Go to http://www-history.mcs.st-andrews.ac.uk/Indexes/Greeks.html and under the column on the left titled Articles About Greek Mathematics you will find listed links to

1. Squaring the Circle
2. Doubling the Cube
3. Trisecting the Angle

In his book A History of Greek Mathematics Vol. I Sir Thomas Heath mentions that the Greeks classified problems by the methods employed to solve them. Problems were planar if they could be solved by constructions involving only straight-edge and compass. Solid problems could be solved using methods that involved conic sections; that is curves obtained from slicing cones (e.g. ellipses, hyperbola etc.). Problems were linear if they involved more complicated curves like spirals etc. Thus solutions could be obtained using planar, solid, or linear methods.

Non-planar Solutions to the Three Classical Problems

A Hippocrates’ Solution to Doubling the Cube

Greek mathematicians were fascinated with ratios between numbers. Three are worth mentioning: the arithmetic, the geometric, and the harmonic mean each of which expresses a relationship between three values.

Given three numbers \( a > x > b \)

Arithmetic Mean is \( a - x = x - b \) or \( x = \frac{a+b}{2} \)

\[ \begin{array}{c}
\hline
a \\
\hline
x \\
\hline
b \\
\hline
\end{array} \]

\[ a - x \quad x - b \]

Geometric Mean is \( \frac{a}{x} = \frac{x}{b} \) or \( x^2 = ab \)

\[ \begin{array}{c}
\hline
a \\
\hline
x \\
\hline
b \\
\hline
\end{array} \]

\[ a \quad x \quad b \]
Harmonic Mean is \( \frac{a-x}{a} = \frac{x-b}{b} \) or \( x = \frac{2ab}{a+b} \)

In the case of the geometric mean, we say \( x \) is a *mean proportion*. Building on this, Hippocrates showed that the *double mean proportion*, that is given numbers \( a \) and \( b \), if the numbers \( x \) and \( y \) were constructed such that

\[
\frac{a}{x} = \frac{x}{y} = \frac{y}{b}
\]

then this would solve the problem of doubling the cube. Assuming that the double mean proportion could be found then eliminating \( x \) yields \( \frac{a}{b} = \frac{y^3}{b^3} \). Letting \( a = 2b \) (i.e. \( a \) is twice length of the base \( b \)) yields \( 2b^3 = y^3 \) showing that \( y \) is the solution.
A solution to trisecting the angle – The Quadratrix of Hippias of Elis

A construction attributed to Hippias called the Quadratrix can be used to trisect an angle.

![Diagram of Quadratrix of Hippias]

Let ABCD be a square with the quarter circle BEGD (in blue) with center A. Let radius AB move uniformly along the arc to position AD. At the same time let parallel B’C’ move uniformly from BC to AD. Let F be the point of intersection of radius AE with parallel B’C’. The locus of points AFIH (in red) obtained from the intersection of AE with B’C’ is the Quadratrix of Hippias. (Note – The center of the square is a point on the Quadratrix - why?)

The Quadratrix easily solves the problem of trisecting an angle (or finding any fractional angle of a given angle). Let $\angle EAD$ be the angle to be trisected. Let AE intersect the Quadratrix at F and let JK be third of the distance of line JF. Construct a line parallel to AD at K. Where this line intersects the Quadratix (at I) is where the angle is trisected! That is $\angle IAD$ is one third $\angle EAD$.

Unfortunately the Quadratrix cannot be constructed using compass and straight-edge alone!
C. Squaring the Circle - Additional properties of the Quadratrix: Where does the Quadratrix intersect AD?

If \( \phi = \angle EAD \) and \((x,y)\) are the coordinates of point F on the Quadratrix, then

\[
y = \tan(\phi)x \quad \text{or} \quad x = \frac{y}{\tan(\phi)} \quad \text{for} \quad \frac{\pi}{2} \geq \phi \geq 0.
\]

Note that by the uniform movements of parallel \( B'C' \) and radius AE that \( \frac{FJ}{AB} = \frac{\phi}{\frac{\pi}{2}} \).

Since \( AB = 1 \), it follows that \( y = \frac{2}{\pi} \phi \). Thus as \( \phi \to 0 \)

\[
\lim_{\phi \to 0} \frac{y}{\tan(\phi)} = \lim_{\phi \to 0} \frac{2}{\pi} \cdot \frac{\phi}{\sin(\phi)} \cdot \cos(\phi) = \frac{2}{\pi}.
\]

So the coordinates of H are \( \left( \frac{2}{\pi}, 0 \right) \).

The fact that we have obtained the length \( \frac{2}{\pi} \) means that we can construct \( \pi \) and once we can construct \( \pi \) (via the Quadratrix) we can square the circle! However this again shows that the Quadratrix is not constructable!