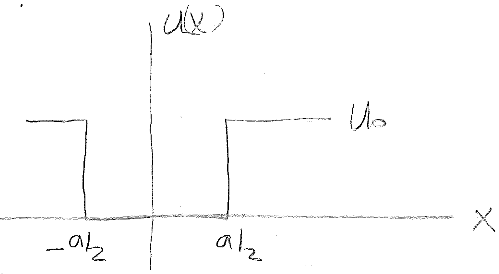
~~7~~ 5. (2.67) finite square well



since $U(x)$ symmetric, $|\psi(x)|^2 = |\psi(-x)|^2$ → symmetric probability distribution

$\Rightarrow \psi(x) = +\psi(-x)$ (symmetric)

or $\psi(x) = -\psi(-x)$ (antisymmetric)

solutions within well $= F\sin kx + G\cos kx$

but $\cos\theta = +\cos(-\theta) \rightarrow$ symmetric

$\sin\theta = -\sin(-\theta) \rightarrow$ antisymmetric

a) symmetric solutions: $(n = 1, 3, 5, \ldots)$

within well: $\psi(x) = G\cos kx \qquad -a/2 < x < a/2$

outside well: $\psi(x) = \overset{0}{\cancel{C}} e^{\alpha x} + D e^{-\alpha x} \qquad x > a/2$ but $C=0$ to keep finite as $x \to +\infty$

$\psi(x) = A e^{\alpha x} + \underset{0}{\cancel{B}} e^{-\alpha x} \qquad x < -a/2$ but $B = 0$ to keep finite as $x \to -\infty$

boundary conditions at $x = -a/2$:

ψ continuous: $A e^{-\alpha a/2} = G\cos(-ka/2) = G\cos(ka/2)$

ψ' continuous: $A\alpha e^{-\alpha a/2} = G(-k)\sin(-ka/2) = Gk\sin(ka/2)$

divide these equations:

$$\frac{\cancel{A} e^{-\alpha a/2}}{A\alpha e^{-\alpha a/2}} = \frac{G\cos(ka/2)}{Gk\sin(ka/2)} \Rightarrow \frac{1}{\alpha} = \frac{1}{k\tan(ka/2)}$$

$\Rightarrow \alpha = k\tan(ka/2)$

or in terms of U_0 & E: $\alpha = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$, $k = \sqrt{\frac{2mE}{\hbar^2}}$

$$\sqrt{\frac{2m(U_0-E)}{\hbar^2}} = \sqrt{\frac{2mE}{\hbar^2}}\tan\left(\sqrt{\frac{2mE}{\hbar^2}}\frac{a}{2}\right) \rightarrow \sqrt{U_0-E} = \sqrt{E}\tan\left(\sqrt{\frac{2mE}{\hbar^2}}\frac{a}{2}\right)$$

b) as $U_0 \to \infty$, $k\tan(ka/2) \to \infty$

or $\tan(ka/2) \to \infty$ $\qquad \tan = \sin/\cos$

$\Rightarrow \frac{ka}{2} \to \pi/2, 3\pi/2, \ldots \Rightarrow k = \frac{n\pi}{a} \qquad n = 1, 3, 5, \ldots$

$\Rightarrow E = \frac{\hbar^2 k^2}{2m} = n^2\frac{\hbar^2\pi^2}{2ma^2}$ as for infinite well ✓

c) $U_0 = 3E_1(\infty) = \frac{3\pi^2\hbar^2}{2ma^2}$; find E_1 (ground-state energy)

we expect $E_1 < E_1(\infty) = \frac{\pi^2\hbar^2}{2ma^2}$; so $E_1 = z\left(\frac{\pi^2\hbar^2}{2ma^2}\right)$ (z between 0+1)

$= zU_0/3$