Using Riemann Sums to Find the Volume of a Cone

A Simple Example

Consider a cone with height 3 and radius 1. What is its volume?

If we approximate the cone with a pile of concentric disks we can estimate the volume of the cone since the volume of a disk (or cylinder) is easy to compute (i.e. \( V_{disk} = \pi r^2 h \))

For example: we can approximate the volume of the cone above using six disks of height \( \frac{1}{2} \) and radii \( \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \) and \( \frac{1}{6} \) inches respectively (use simple proportions to figure this out). Using the formula for the volume of a disk six times and a little arithmetic yields
\( \frac{91}{72} \pi \approx 3.970624048 \) for the sum of the volumes of the six disks which approximates the volumes of the cone.

**Exercise:**

Verify the above calculation. Is this estimate an upper bound or a lower bound for the actual volume of the cone?

Find a lower bound estimate for the volume of the cone. Hint: use inscribed disks (one of which has a zero radius).

**Using a Riemann Sum to Approximate the Volume**

Without too much difficulty we can generalize the above to estimate the volume of any cone with arbitrary height \( h \) and radius \( r \). Take the x-y plane. Tilt the cone on its side, aligning the main axis of the cone the with positive x-axis, placing the apex at the origin \((0,0)\) and positioning the center of the base at \((0, h)\). The projection of the cone into the x-y plane appears as a triangle with the upper corner at coordinates \((h, r)\). Notice that disks as projected onto the plane appear as rectangles.

With this set up we can easily obtain a general formula for the sum of the volumes of any number of disks which in turn approximates the volume of the cone.

Begin by partitioning the x-interval \([0, h]\), the length of cone’s axis, into \( n \) equal sub-intervals. Each sub-interval will have length \( \frac{h}{n} = \Delta x \) so \( \Delta x \) is the width or thickness of each disk.
The radius of each disk is obtained by using the equation of the line determined by the two points (0,0) and (h,r) which is \( y = \frac{r}{h} x \) (remembering that \( h \) and \( r \) are constants). The radius of the first disk is \( \frac{r}{h} \Delta x \). The radius of the 2\(^{nd} \) disk is \( \frac{r}{h} 2 \Delta x \). And the radius of the \( n^{th} \) or last disk is \( \frac{r}{h} n \cdot \Delta x \). So the sum of the volumes of all \( n \) disks is

\[
\pi \left( \frac{r}{h} \Delta x \right)^2 \Delta x + \pi \left( \frac{r}{h} 2 \Delta x \right)^2 \Delta x + \pi \left( \frac{r}{h} 3 \Delta x \right)^2 \Delta x + \ldots + \pi \left( \frac{r}{h} n \cdot \Delta x \right)^2 \Delta x
\]

This is a Riemann Sum!

Factoring out common terms like \( \pi \), \( \frac{r}{h} \), and \( \Delta x \), recalling that \( \Delta x = \frac{h}{n} \), and doing some algebra simplifies the Riemann Sum to

\[
\pi \left( \frac{r}{h} \right)^2 \Delta x \left( \frac{h}{n} \right)^3 \left( 1^2 + 2^2 + 3^2 + \ldots + n^2 \right) = \pi \left( \frac{r}{h} \right)^2 \left( \frac{h}{n} \right)^3 \left( 1^2 + 2^2 + 3^2 + \ldots + n^2 \right) = \pi \cdot \frac{r^2 h}{n^3} \left( 1^2 + 2^2 + 3^2 + \ldots + n^2 \right)
\]

Now it can be shown that the open expression \( 1^2 + 2^2 + 3^2 + \ldots + n^2 \) for the sum of the first \( n \) squares is equal to the closed form \( \frac{n \cdot (n+1) \cdot (2n+1)}{6} \). Thus

\[
\pi \cdot \frac{r^2 h}{n^3} \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6} = \pi \frac{r^2 h}{6} \frac{(n+1)(2n+1)}{n^2}
\]

**Reality Check:** If \( r = 1 \), \( h = 3 \) and \( n = 6 \) (the values used for the cone at the beginning of this paper), verify that you get the same answer - \( \frac{91}{72} \pi \).

Since the above calculations used disks that covered the cone (circumscribed disks) the Riemann Sum approximation of the volume obtained is an upper bound meaning the real volume of the cone is actually smaller. We can used disks that fit inside the cone (the inscribed case) where the radius is determined by the left hand sides of the inscribed rectangles (the dotted rectangles). It’s not difficult to obtain the Riemann Sum given below which is a lower bound estimate for the volume of a cone (note that the left most inscribed disk has radius 0).
Hence putting the lower and upper bound Riemann Sums together we know
\[
\pi \cdot \frac{r^2 h}{n^3} \cdot \frac{(n-1)}{6} \cdot \frac{(2n+1)}{6n^2} \leq \text{Volume}_{\text{cone}} \leq \pi \cdot \frac{r^2 h}{n^3} \cdot \frac{(n+1)}{6n^2} \frac{(2n-1)}{6n^2}
\]

The Limit of the Riemann Sum.

What happens as the number of rectangles (or disks) used to approximate the volume increases (without bound). Intuitively we observe that we get better and better approximations to both the lower and upper bounds for the volume. But specifically what happens in the limit?

\[
\lim_{n \to \infty} \pi r^2 h \cdot \frac{(n+1)(2n+1)}{6n^2} = \pi r^2 h \cdot \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2} \]

Since

\[
\lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n \to \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \to \infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} = \frac{1}{3}
\]

and likewise

\[
\lim_{n \to \infty} \frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}
\]

Both limits are the same! It follows that since the limits of both the upper and lower bounds (which sandwich the volume of the cone) approach the same value, the true volume of the cone is

\[
\text{Volume}_{\text{cone}} = \frac{1}{3} \pi r^2 h
\]

Thus the formula for the volume of a cone!